MU120 PRBB

The Open University

Mathematics and Computing
A first level multidisciplinary course

PREPARATORY RESOURCE BOOK B

Modules 5 – 7

Open Mathematics

RB

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Contents

How to use this book

This is the second and last of the *Preparatory Resource Books*. It contains Modules 5–7.

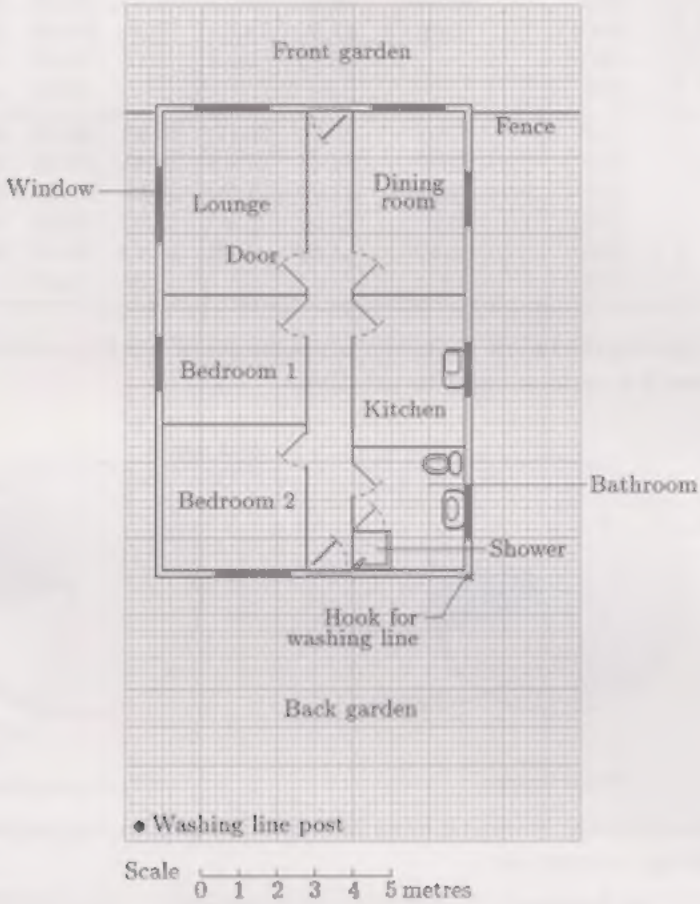
You should use this book in the same way that you did *Book A*. Reread the instructions at the start of that book if you are unsure of how to proceed.

Remember that this is a *resource* book, not a study text; it is *not* intended that you work through this material from beginning to end.

Module 5 Diagrams, charts and graphs

Try these first

1 Below is a scale plan of a new bungalow and its garden.



(a) How wide is the back garden?

(b) What are the dimensions of the lounge?

(c) Wall units (of full room height) come in sections 1 m wide, 0.3 m deep. How many can be fitted into the lounge without blocking the windows or door? Draw a larger scale plan of the lounge to show how these wall units can be arranged.

(d) Is the bathroom big enough for a 1.8 m by 0.8 m bath?

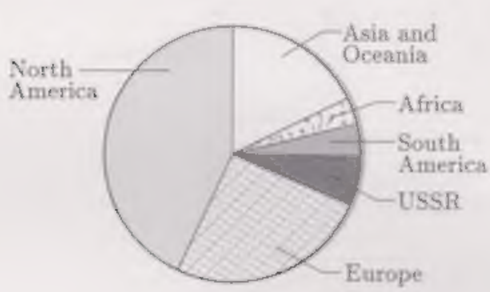
(e) If a washing line is to be run from the washing line post to the hook on the corner of the house outside the bathroom, how long will it be?

2 On a scale plan of a garden, where 2 cm represents 1 m, a lawn is represented by a 4 cm by 7 cm rectangle. What is the actual size of the lawn?

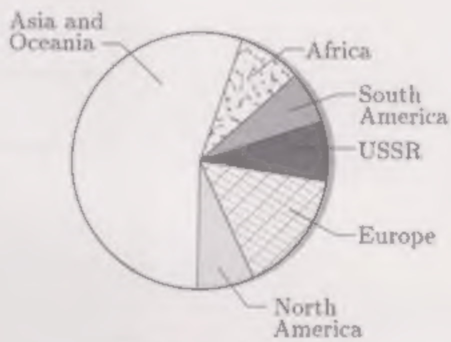
3 The table below shows rent rebates and allowances for different categories of families. How much allowance could a family with two dependent children and a weekly income of £275 claim, if their rent was £45 a week?

Family size	Weekly income (£)	Weekly rent (after taking off rates, etc.) £20 or less	£25	£30	£35	£40	£45	£50
		Rent rebate or allowance (£)						
Couple (or single parent) and one dependent child	175	20.00	25.00	30.00	34.75	37.75	40.75	43.75
	200	19.50	22.50	25.50	28.50	31.50	34.50	37.50
	225	13.25	16.25	19.25	22.25	25.25	28.25	31.25
	250	8.60	11.60	14.60	17.60	20.60	23.60	26.60
	300	—	3.10	6.10	9.10	12.10	15.10	18.10
	350	—	—	—	—	3.60	6.60	9.60
Couple (or single parent) and two dependent children	200	20.00	25.00	30.00	35.00	39.70	42.70	45.70
	225	20.00	24.45	27.45	30.45	33.45	36.45	39.45
	250	15.20	18.20	21.20	24.20	27.20	30.20	33.20
	275	9.90	12.90	15.90	18.90	21.90	24.90	27.90
	325	1.40	4.40	7.40	10.40	13.40	16.40	19.40
	375	—	—	—	1.90	4.90	7.90	10.90

4 The pie charts below show the proportions of world population and income for various areas in the 1980s.



World income



World population

Approximately what fractions of world income and population do the following account for:

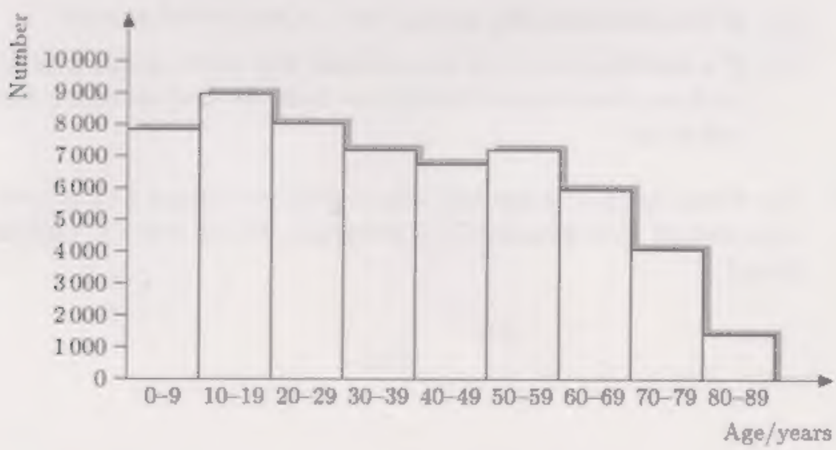
(a) North America (b) Asia and Oceania (c) USSR

(d) Western civilizations (Europe and North America)

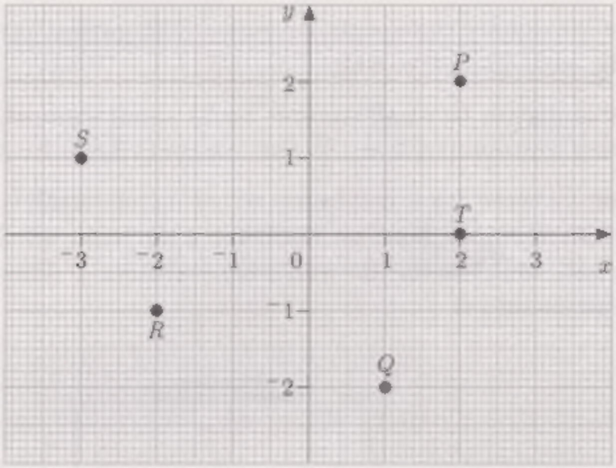
5 The histogram below shows the numbers of people in different age groups in a sample of the UK population.

(a) What is the width of each age group?

(b) Which age group contains the largest number of people?



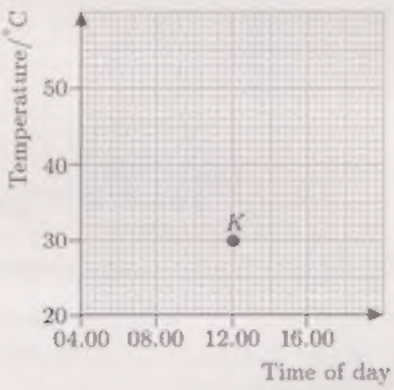
6 Consider the following diagram:



(a) Write down the coordinates of the points P, Q, R, S and T.

(b) On the same diagram, plot B $(^-1.5, 1.2)$, C $(^-2.8, ^-1.8)$ and D $(0, 2.2)$.

7 Write down the coordinates of the point K shown below, and interpret these coordinates in terms of the labelling of the axes.

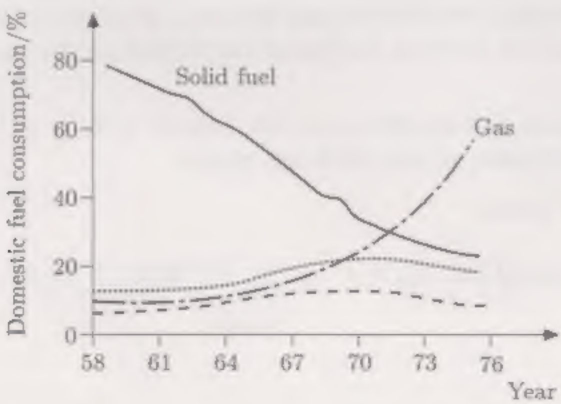


8 Plot the following points on graph paper: $(14, 0)$, $(0, 50)$, $(^-18, 56)$, $(20, 48)$.

9 Draw a graph based on the following data. What relationship is suggested?

Length of bus journey in km	0.5	1	1.5	2	2.5	3	3.5	4
Cost in £	0.2	0.28	0.32	0.42	0.50	0.62	0.69	0.86

10 The graphs below illustrate domestic fuel consumption. Use the graphs to compare solid fuel consumption and gas consumption during the period 1958 to 1976.

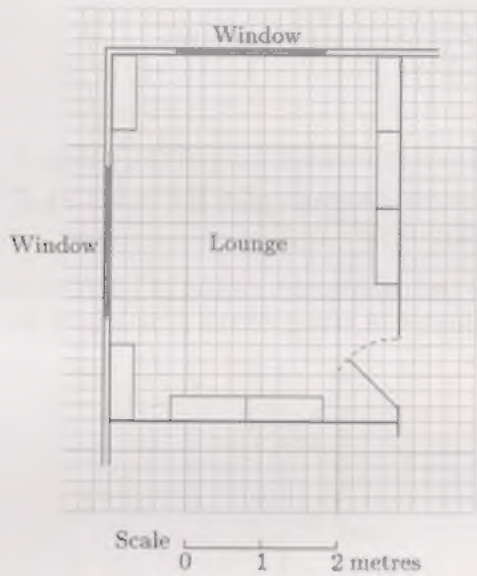


Check your answers

Section 5.1 ***1*** (a) One big square represents 2 metres. The garden is 6 big squares wide, so is 12 metres wide.

(b) The lounge is 19 small squares by 24 small squares. Each small square represents 0.2 m. So the lounge is 3.8 m by 4.8 m.

(c) There is room for a maximum of seven: one each on either side of the side window; none on the front wall; three on the wall next to the hall; and two on the wall with bedroom 1. One possible arrangement is:



(d) Yes, along the back wall next to the shower.

(e) One metre is represented by 5 mm on the scale plan. The post and the hook are 55 mm apart on the scale plan. So the line will be $55 \div 5 = 11$ metres long.

Section 5.1 ***2*** 2 cm represent 1 m, so $4 = 2 \times 2$ cm represents 2 m and $7 = \frac{7}{2} \times 2$ cm represents $\frac{7}{2}$ m or 3.5 m. Hence, the lawn is 2 m by 3.5 m.

Section 5.2.1 ***3*** £24.90.

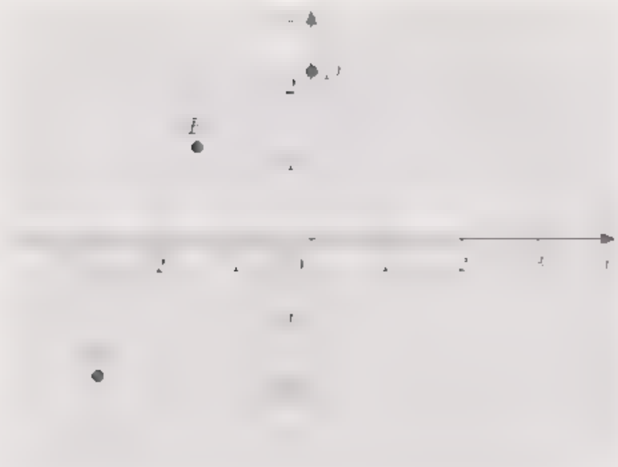
Section 5.2.2 ***4*** North America accounts for almost half the world's income but only about a twelfth of the total population. Asia and Oceania account for over half of the total population but only about a fifth of the world's income. The USSR accounts for about the same proportion (about a twelfth) of population and income. Western civilizations (Europe and North America) contribute only about a quarter of the world's population but account for about two-thirds of the world's income.

Section 5.2.3 ***5*** (a) 10 years. For example, the ten ages 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 years make up the first age group.

(b) 10–19 years.

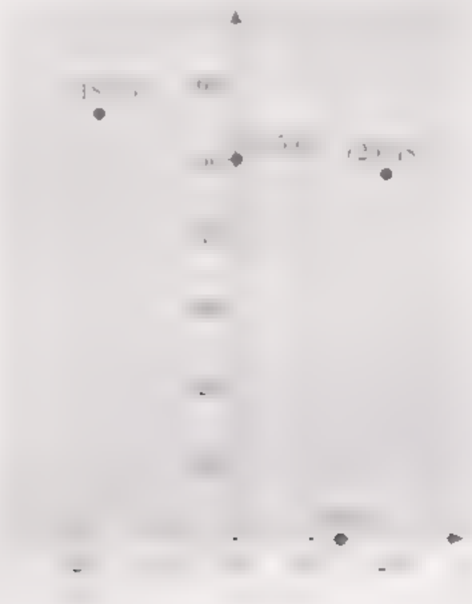
Section 5.3.1 ***6*** (a) P (2, 2), Q (1, $^-2$), R ($^-2$, $^-1$), S ($^-3$, 1), T (2, 0).

(b)

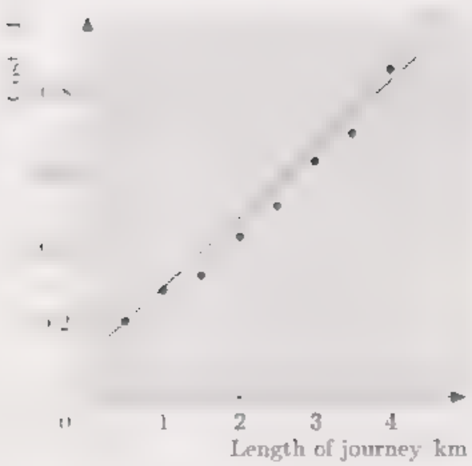


7 *K* has coordinates (12.00, 30). This tells us that the temperature at midday was 30 °C. Section 5.3.1

8 Your answer depends on the size of the graph paper and the scales you used, but it should look something like the following: Section 5.3.1



9 Section 5.3.2

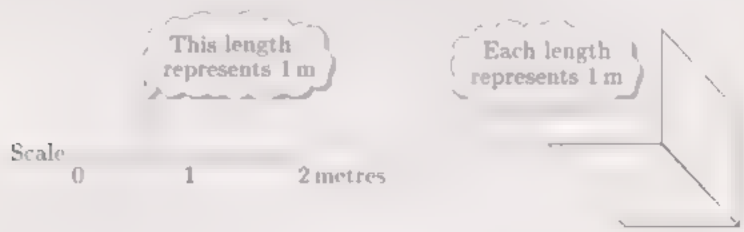


The graph is approximately a straight line. It indicates that the cost of a bus journey increases roughly in proportion to its length.

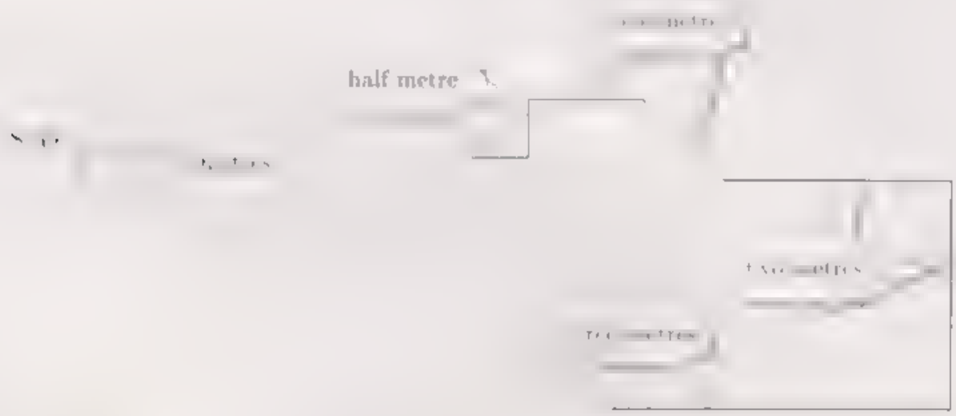
Section 5.3.2 **10** Solid fuel consumption decreased steadily between 1958 and 1976. Gas consumption increased slowly until 1967 then more rapidly in the 1970s. (This was probably due to the introduction of natural gas.) Whereas solid fuel accounted for about 80% of domestic fuel consumption in 1958, in 1976 it accounted for only about 25% of the market. On the other hand, gas jumped from about 10% to nearly 60% of the market in this period.

5.1 Scale diagrams

Plans of houses and instructions for assembling shelves, etc. often come in the form of **scale diagrams**. Each length on the diagram represents a length in the real house, the real shelves, etc. Usually a scale is given on the diagram, so that you can see which length on the diagram represents a standard length, such as a metre, in the real object. This length always represents the *same* standard length, wherever it is on the diagram and in whatever direction.

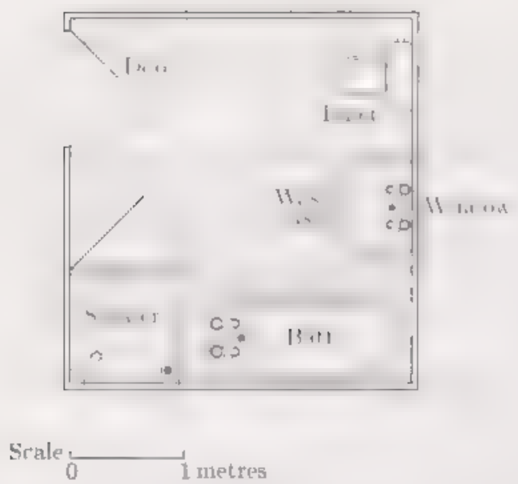


Other lengths may represent fractions or multiples of this standard length. For example, lengths which are half as long on the diagram represent lengths which are half as long in reality; lengths which are twice or three times as long on the diagram represent lengths which are two or three times as long in reality.



Example 1

Below is a scale plan of a bathroom. What are the dimensions of the bathroom, the shower and the bath? What is the width of the door?



The background squares show the length representing 1 metre.

Solution

The top and bottom walls in the diagram are 3 squares wide, and so the bathroom is 3 metres wide. The side walls in the diagram are 3 and a bit squares long. If you measure the 'bit' you will find that it is $\frac{1}{5}$ of the length representing one metre and so it represents $\frac{1}{5}$ of a metre or 0.2 metres. So the total length is 3.2 m. Hence the bathroom is 3 m by 3.2 m.

The shower in the diagram is 1 square in each direction, so in reality it is 1 m by 1 m.

The bath in the diagram is nearly 2 squares long. If you measure it you will find it is 1 square plus $\frac{4}{5}$ or 0.8 of a square long. It is also $\frac{4}{5}$ or 0.8 of a square wide in the diagram. So in reality its dimensions are 1.8 m by 0.8 m.

The door is 1 m wide—it is the same width as the length on the scale that represents 1 metre.

Example 2

If the scale on a diagram is such that 2 cm represents 1 m, what lengths do 4 cm, 0.2 cm, 3 cm, 3.6 cm and 0.5 cm represent?

Solution

4 cm is 2 times 2 cm so represents 2 m.
0.2 cm is $\frac{1}{10}$ of 2 cm so represents $\frac{1}{10}$ m or 0.1 m.
3 cm is 1.5 times 2 cm so represents 1.5 m.
3.6 cm is 1.8 times 2 cm so represents 1.8 m.
0.5 cm is $\frac{1}{4}$ of 2 cm so represents $\frac{1}{4}$ m or 0.25 m.

$0.2 \div 2 = \frac{1}{10}$

Try some yourself (5.1)

Solutions on page 85

1 On the diagram of the bathroom in Example 1, what is the width of the window and what are the dimensions of the wash basin?

2 The scale on a diagram is such that 5 cm represents 1 m. What lengths do the following represent: 10 cm, 20 cm, 1 cm?

3 On a diagram of a new town 2 cm represents 1 km. What lengths on the diagram represent 10 km, 5 km, 0.5 km?

4 Draw a scale plan of the following garden, using a scale in which 0.5 cm represents 1 m:

> The garden is 10 m by 20 m. It has a flower bed 2 m wide along the whole of one of the long and both of the short sides, and a 1.5 m wide path along the rest of the other side. Another 1.5 m wide path joins this path in a T junction and leads to a sundial at the centre of the garden.

5.2 Tables and charts

5.2.1 Tables

Data is the plural of datum, which is the Latin word for 'given'.

Experiments or surveys usually generate a lot of information from which to draw conclusions. Such information is called *data*. Data can also be extracted from newspapers or books.

One convenient way to present data is in a **table**. Sometimes information can be recorded in a table as it is collected. For example, many motorists keep a log of their car's petrol consumption, such as that shown below, in which the information is recorded as soon as it is collected.

Date	Litres	Mileage
4.1 95	49.4	22 695
10.1.95	49.1	22 916
20.1.95	48 6	23 212
24.1.95	49 2	23 320
27.1.95	48 5	23 595

Experiments often require regular measurements. Again, information can be recorded as it is collected. For example, the table below resulted from an experiment to determine how quickly a cup of tea cooled down.

Time (mins)	0	5	10	15	20	25	30	35	40	45	50
Temperature (°C)	90	65	60	50	36	35	30	26	25	22	20

Tables can be laid out vertically (as in the petrol log) or horizontally (as in the tea experiment). Each column or row heading should indicate what is being measured and the unit of measurement.

Conclusions can often be drawn directly from tables.

Example 3

Jane travels to work by car. The table indicates her journey times over a two-week period.

		Mon	Tue	Wed	Thu	Fri
Time	Week 1	14	16	20	13	15
(mins)	Week 2	15	17	14	10	17

(a) What was the shortest journey time?

(b) What was the longest journey time?

(c) Suggest reasons for such a wide range of journey times.

Solution

(a) Reading across the table, the lowest number is 10. So the shortest journey time was 10 minutes.

This occurred on the second Thursday.

(b) Similarly, the longest journey time was 20 minutes.

This was on the first Wednesday.

(c) Possible reasons why Jane's journey times vary as they do include traffic jams and the use of different routes. Also, if her working hours are not regular, the variation could be caused by her travelling at different times of day. (Notice that we only *suggest* reasons. Without any other information we cannot say anything definite.)

You can *state* factual conclusions, but only *suggest* reasons. Interpretation of the information often depends on your own experience or some other information not included in the table.

Example 4

The table below is taken from a holiday brochure.

Inclusive prices per person in £s (excluding airport taxes)

Hotel	**Pueblo** Full board									
Departure airport	GATWICK		LUTON		BIRMINGHAM		MANCHESTER		GLASGOW	
No. of nights in hotel	7	14	7	14	7	14	7	14	7	14
Aircraft	B707	B707	B720	B720	B737	B737	B727	B727	B727	B727
Approx. dep. time	1530	1530	0810	0810	0845	0845	0840	0840	1645	1635
Approx. arr. back time	2120	2120	1355	1355	1455	1455	1435	1435	2320	2320
First departure	26 Apr	26 Apr	26 Apr	26 Apr	26 Apr	26 Apr	26 Apr	26 Apr	17 May	17 May
Last departure	28 Oct	[illegible] Oct	18 Oct	11 Oct	28 Oct	[illegible] Oct	28 Oct	11 Oct	27 Sep	[illegible] Sep
Holiday number	T2255									
Dep. between										
26 Apr	980	1330	1010	1360	1030	1380	1080	1430	—	—
28 Apr–5 May	1240	1580	1270	[illegible]	[illegible]	1630	[illegible]	1680		
6–18 May	1200	1540	1230	1570	1250	1590	1300	1640	1450	1790
19–26 May	1300	1740	1330	1770	[illegible]	1790	1400	1840	1750	1990
27 May–5 June	1370	1820	1400	1850	1420	1870	1470	1920	1620	2070
[illegible] June–13 July	1470	[illegible]	[illegible]	2020	1520	2040	1570	2090	1720	2240
14 July–5 Aug	1530	2070	[illegible]	[illegible]	[illegible]	[illegible]	1630	2170	1780	2320
6–31 Aug	1570	2110	[illegible]	[illegible]	[illegible]	[illegible]	1670	2210	[illegible]	[illegible]
1–14 Sep	1510	2040	1540	2070	1560	2090	1610	2140	1760	2290
15–28 Sep	[illegible]	1870	[illegible]	1900	[illegible]	[illegible]	[illegible]	1970	[illegible]	2120
29 Sep–28 Oct	1150	[illegible]	1180	1520	[illegible]	1540	1250	1590		
Children's reduction 2–11 years inc.										
23 Apr–15 June & from 1 Sept	FREE 50% when allocation of free holidays full									
16 June–31 Aug	20%									
Approx. transfer time airport/hotel	1 hr 5 mins									

Flying time — Gatwick/Alicante 2 hrs 10 mins; Luton/Alicante 2 hrs 15 mins; Birmingham/Alicante 2 hrs 20 mins; Manchester/Alicante 2 hrs.

The Robertsons have two children, Tim aged 8 and Helen aged 11. They are leaving from Luton for the Hotel Pueblo on 6 August for two weeks. What information does the table give about their holiday?

Solution

We need to identify the relevant parts of the table.

Since the Robertsons are leaving from Luton and staying for two weeks, we need to look at the column for departure airport Luton and for 14 nights in the hotel, as shown below. The parts of that column relevant to the Robertsons are picked out in bold.

Departure airport	**LUTON**	
No. of nights in hotel		**14**
Aircraft		**B720**
Approx. dep. time		**0810**
Approx. arr. back time		**1355**
First departure		26 Apr
Last departure		11 Oct
Holiday number	**T2255**	
Dep. between		
26 Apr		1360
28 Apr 5 May		1610
6–18 May		1570
19–26 May		1770
27 May–15 June		1850
16 June–13 July		2020
14 July 5 Aug		2100
6–31 Aug		**2140**
1 14 Sep		2070
15–28 Sep		1900
29 Sep–18 Oct		1520

Other relevant information can be found in the Children's reduction, Approx. transfer time and Flying time rows at the bottom of the table.

Since both children are in the '2–11 years inclusive' category, they qualify for 20% reductions from 16 June till 31 August.

So the Robertsons fly on aircraft B720 leaving at approximately 08.10 on 6 August and returning at about 13.55 two weeks later. The holiday number is T2255. The holiday costs £2140 for each adult with a 20% reduction for each child, giving a total of £7704. The flight time from Luton to Alicante is 2 hours 15 minutes and it takes about 1 hour and 5 minutes to transfer from the airport to the hotel. (That's quite a lot of information from one table!)

Tables often give information in percentages. The table below indicates the results of a survey of household sizes in Great Britain in the 1970s.

Table 1 Household sizes in Great Britain 1971 to 1977

No. of persons in household (all ages)	1971 (%)	1973 (%)	1975 (%)	1976 (%)	1977 (%)
1	17	19	20	21	21
2	31	32	32	32	33
3	19	18	18	17	17
4	18	18	17	17	18
5	8	8	8	8	7
6 or more	6	5	5	5	4
Base (households) 100%	11 988	11 651	12 097	12 120	11 979

For example, in 1973 19% of households surveyed comprised only one person, 32% consisted of two people, 18% consisted of three people and so on. In 1973 the actual number of households surveyed consisting of two people is given by calculating 32% of 11 651, which is 3728.32, or 3728 rounded to the nearest whole number (we can only really count whole numbers of households).

The figures for 1973 were calculated from a total number (a *base*) of 11 651 households.

Since each column should include all the surveyed households, the total of all percentages in the column should be 100%; and indeed, for 1973,

$$19 + 32 + 18 + 18 + 8 + 5 = 100.$$

However, the column total is not always exactly equal to 100%. All the percentages have been rounded to whole numbers and this can sometimes introduce rounding errors. For example, the total of 1971 column is

$$17 + 31 + 19 + 18 + 8 + 6 = 99$$

Rounding errors are usually very small, so that the total is always very close to 100%.

Sometimes the total percentages for both rows and columns are indicated. The table below indicates the percentages of families with different numbers of dependent children.

Table 2 Families with dependent children

Type of family		Number of dependent children				
		1 (%)	2 (%)	3 (%)	4 or more (%)	Total (%)
Married couple	(%)	32.8	34.9	15.4	8.2	91.3
Lone mother	(%)	3.9	2.1	1.1	0.6	7.7
Lone father	(%)	0.8	0.2	0	0.1	1.1
Total	(%)	37.5	37.2	16.5	8.9	100

Base = 100% = 4835

In examples like this the row totals *and* the column totals should add up to 100%, although rounding errors might mean that the total is not exactly equal to 100%. In this case the row totals and the column totals both add up to 100.1% (the sum of the row totals and the sum of the column totals must *always* be the same).

$91.3 + 7.7 + 1.1 = 100.1$
$37.5 + 37.2 + 16.5 + 8.9 = 100.1$

Try some yourself (5.2.1)

Solutions on page 85

1 The table below indicates the cooling rate of tea.

Time [illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Temperature (°C)	90	65	60	50	36	35	30	26	25	22	20

(a) How long does it take for the tea to cool to 50 °C?

(b) By how much does the temperature drop in the first twenty minutes? By how much in the second twenty minutes?

(c) What can you suggest about the pattern of the cooling?

2 (a) Paul and Linda Potter wish to take their 12-year-old daughter to the Hotel Pueblo for a week's holiday, leaving from Birmingham on 6 October. What information is given in the table in Example 4 about such a holiday?

(b) Which is the most expensive holiday in the table? Which is the cheapest?

3 Consider Table 1 on household sizes above.

(a) What was the 100% base for the 1977 survey?

(b) How many of the households surveyed in 1977 consisted of four people?

(c) What is the percentage total of the 1977 column?

4 Use Table 2 on families with dependent children above to calculate the actual number of families consisting of a married couple with two dependent children.

5 The table below shows the cigarette smoking habits in various age groups of a sample of men.

Table 3 Cigarette smoking habits of men

Age		No. of cigarettes smoked per day			
		none	under 20	over 20	total (= 100%)
16–24	(%)	62	18	19	1850
25–34	(%)	52	19	29	2560
35–49	(%)	52	20	28	2470
50–59	(%)	50	21	28	1960
60 or over	(%)	60	24	16	2150
all aged over 16	(%)	55	22	23	10990

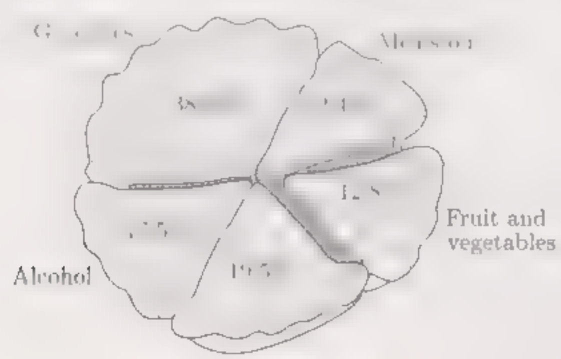
(a) Write down the percentage of male smokers aged between 25 and 34 who smoke under 20 cigarettes a day.

(b) How many of the men aged 60 or over 60 are non-smokers?

(c) Which age groups contain the heaviest smokers?

5.2.2 Pie charts

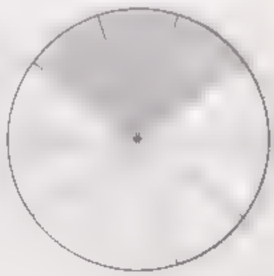
Pie charts are pictures that enable us to compare proportions. In particular they allow quick identification of very large proportions and very small proportions, and are generally based on large sets of data.



The pie chart above summarizes weekly expenditure on food and drink. The whole pie represents 100% of the expenditure. The pie is then broken up into 'slices' and the area of each 'slice' represents a fraction or percentage of the total expenditure. For example, groceries account for 38.0% of the total expenditure. The area of this 'slice' is 38/100 of the total area.

In this example, because of rounding errors, the percentages actually add up to 100.2%

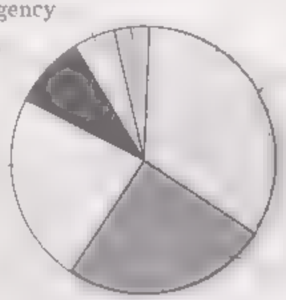
Pie charts can be constructed by dividing a circle into equal 'slices' and then shading the appropriate fractions. For example, the following circle is divided into 10 equal 'slices'. The shaded area is $\frac{3}{10}$ or 30% of the total.



Unless percentages or fractions are indicated on the diagram, it is difficult to glean any precise information from a pie chart, although the percentages and fractions *can* be estimated.

Example 5

This pie chart indicates how people first heard about their present job. Interpret what the pie chart indicates by estimating the percentages of people in each category.



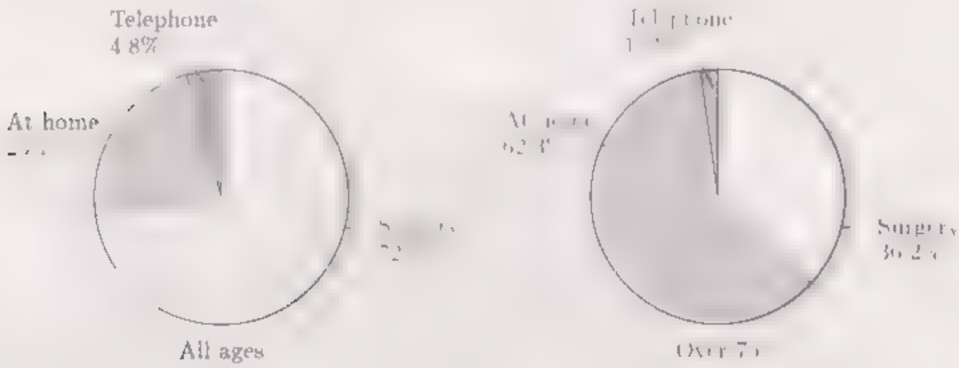
Source of first hearing about present job

Solution

The largest proportion, about 35%, heard about their job from a relative or a friend. About 25% took the initiative from an advertisement and about another 25% from a direct approach. Only about 15% heard of their job through an employment exchange or private agency.

The 'other' category covers all other sources of information.

Pie charts give a good visual comparison of proportions arising from different sets of similar data. The pie charts below summarize the means by which people consult their GP. The pie chart on the left summarizes the information for people of all ages; the pie chart on the right summarizes the information for elderly people. Taking the population as a whole, most people (72.1%) consult their doctor at his surgery. However, most elderly people (62.3%) are likely to be visited at home.

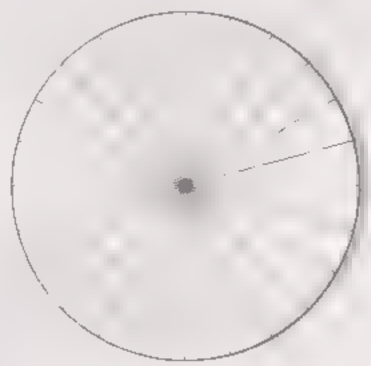


Solutions on page 86

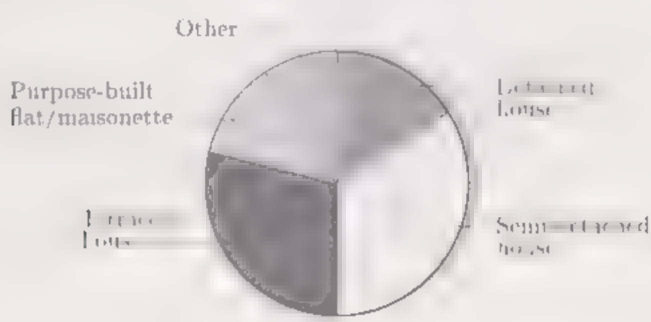
Try some yourself (5.2.2)

1 The table below breaks up Tom's activities for the day. Shade in the pie chart to illustrate how the day is broken up. Label each category on the chart.

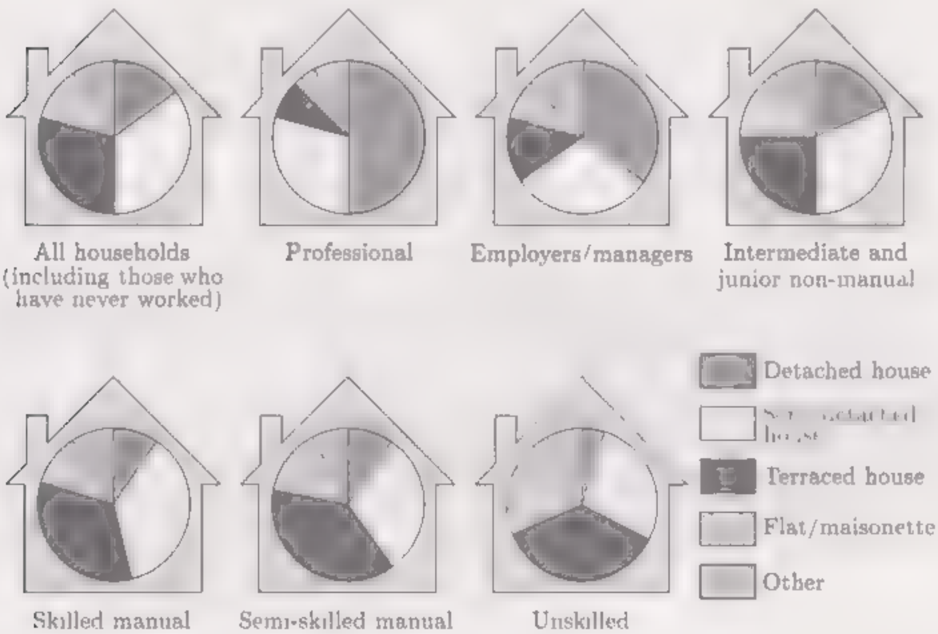
Activity	Time (hours)
Sleeping	9
Working	7
Eating	3
Watching TV	4
Pub	1
Total	24



2 The pie chart below shows the proportions of people living in different kinds of accommodation. Interpret what the pie chart indicates by estimating the percentages of people in each category.



3 The diagram below shows the proportions of people living in various kinds of accommodation. Separate pie charts are drawn for people in different types of occupation.



(a) Which type of accommodation accounts for the largest proportion of professionals?

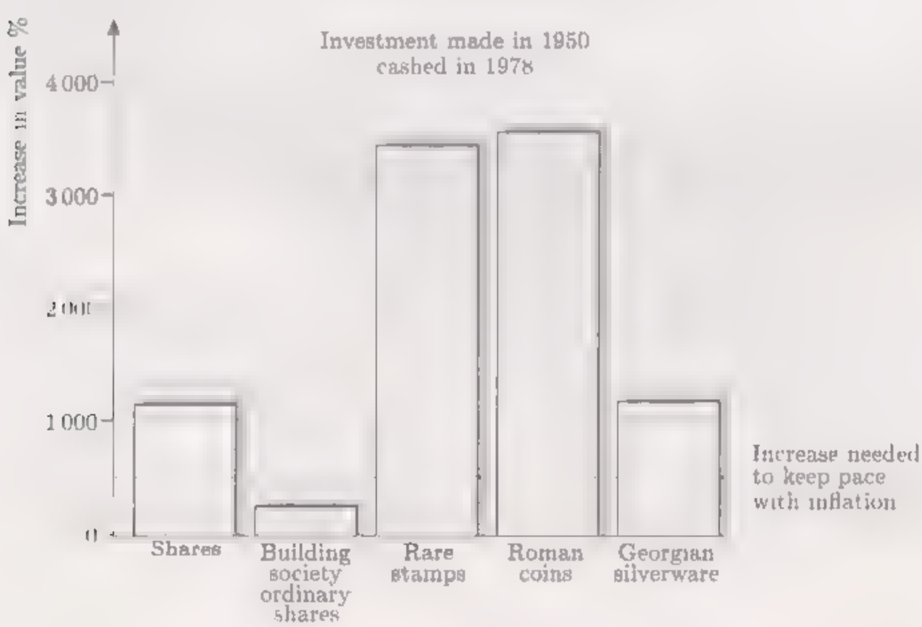
(b) Which type of accommodation accounts for the largest proportion of unskilled workers?

(c) Overall, which is the most common type of accommodation?

(d) Compare the accommodation used by employers/managers with that used by skilled manual workers.

5.2.3 Bar charts and histograms

Bar charts are commonly used to represent data since they allow quick assimilation of the information and immediate comparison.



This bar chart compares investment in 'things' with more conventional forms of investment. The height of each bar or column represents the percentage increase in value between 1950 and 1978. For example, shares increased in value by over 1000%. Roman coins proved to be the best investment since the height of that column is the greatest. They increased in value by over 3500%. Building society shares, with the shortest bar, proved to be the worst investment and did not even keep pace with inflation.

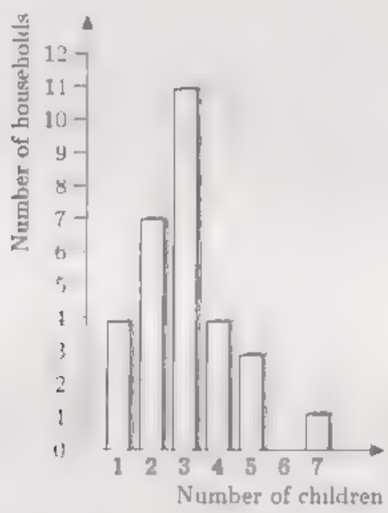
The bars on a bar chart are usually drawn so that they do not touch each other.

Example 6

The table indicates the numbers of children in a sample of 30 households. Draw a bar chart to illustrate the information.

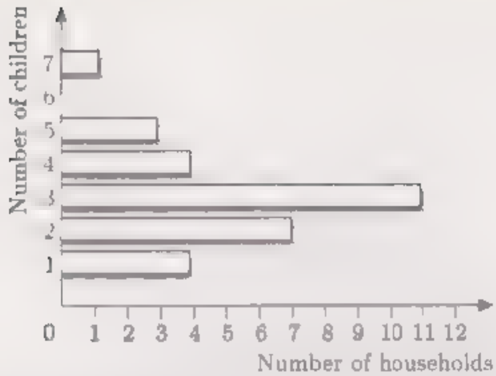
Number of children	Number of households
1	4
2	7
3	11
4	4
5	3
6	0
7	1
Total	30

Solution



The number of children is measured along the horizontal axis and the number of households up the vertical axis. The height of each bar represents the number of households containing that number of children.

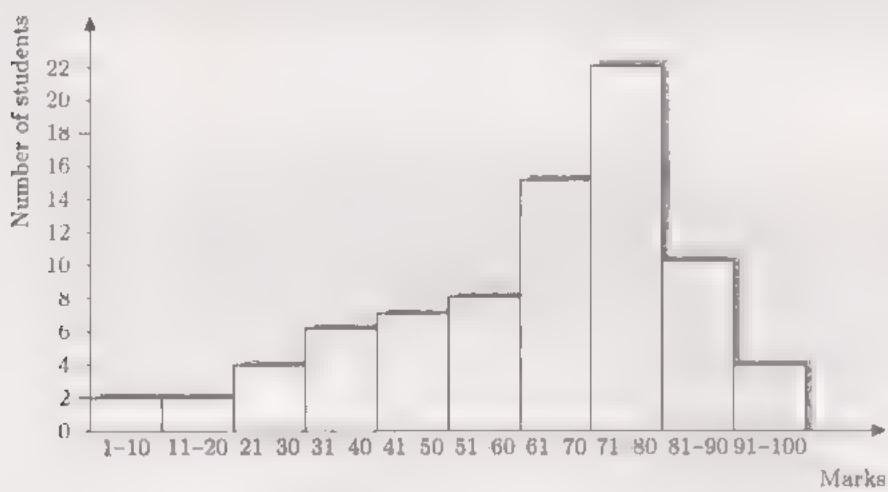
The bars in bar charts can be drawn horizontally as well as vertically. For example, the bar chart in Example 6 could have been drawn as follows.



A **histogram** is a diagram, similar to a bar chart, that provides a visual representation of how data are distributed across a *range* of values. It is constructed in the same way as a bar chart, except that the bars or columns are drawn so that they touch. For example, consider the following table of examination scores.

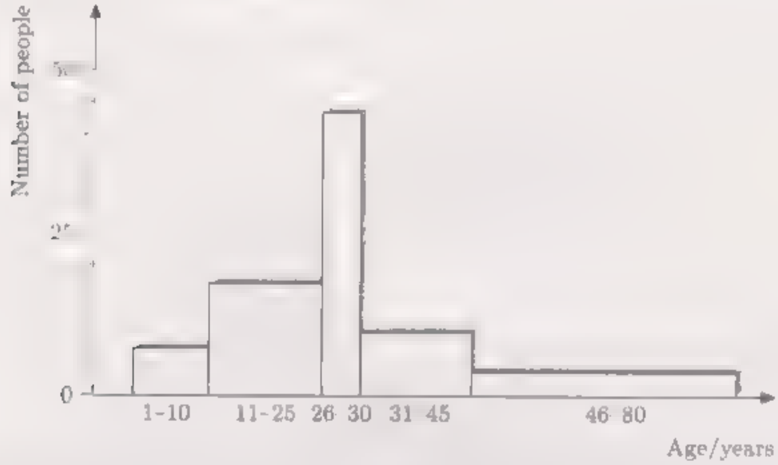
Range of marks	Number of students
1–10	2
11–20	2
21–30	4
31–40	6
41–50	7
51–60	8
61–70	15
71–80	22
81–90	10
91–100	4
Total	80

The following histogram represents the information given in the table.



The horizontal axis is divided into intervals to represent each group of data: 1–10, 11–20, 21–30 and so on. Notice that each mark belongs to only one interval. It's no good dividing the line 1–10, 10–20, 20–30 and so on since then the mark 10, for example, would be included in two intervals: 1–10 and 10–20. The intervals in this histogram are all equal, and the *height* of each bar or column indicates the number of students whose marks fall in the corresponding range.

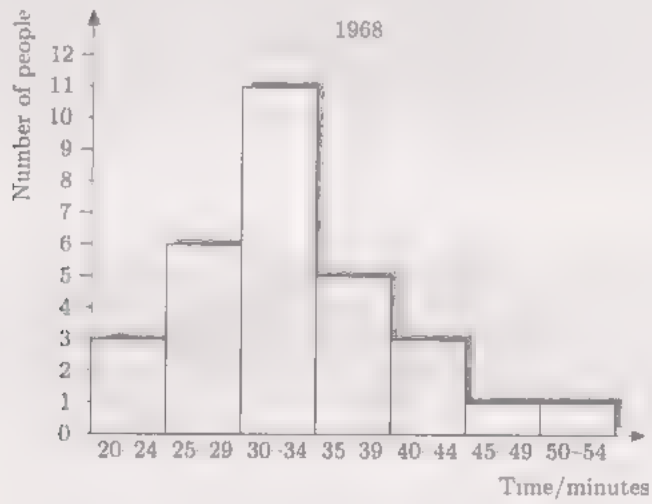
Not all histograms are based on equal intervals. In the histogram below some intervals are wider than others.

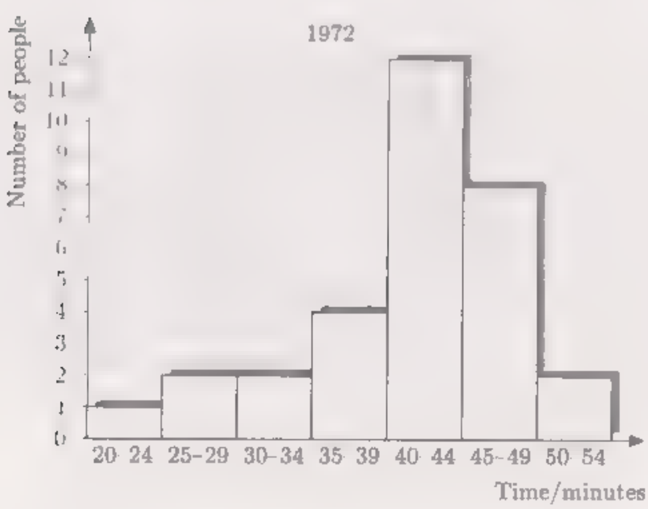


area = height × width

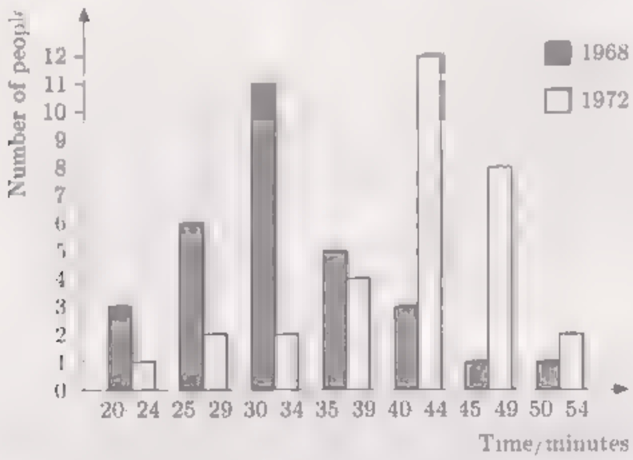
This example is included as a warning. It is very difficult to compare data when the intervals are unequal. You must compare the *area* of each rectangle to make any useful comparisons. However, most histograms are based upon equal intervals. Because the width of each bar or column is the same, the area of each is proportional to its height. Therefore the different categories can be compared by height alone.

Sometimes you may need to compare the data given in two or more bar charts or histograms. The two histograms below resulted from two surveys to investigate the time taken to travel from Central London to Heathrow Airport. The first was in 1968; the second in 1972.



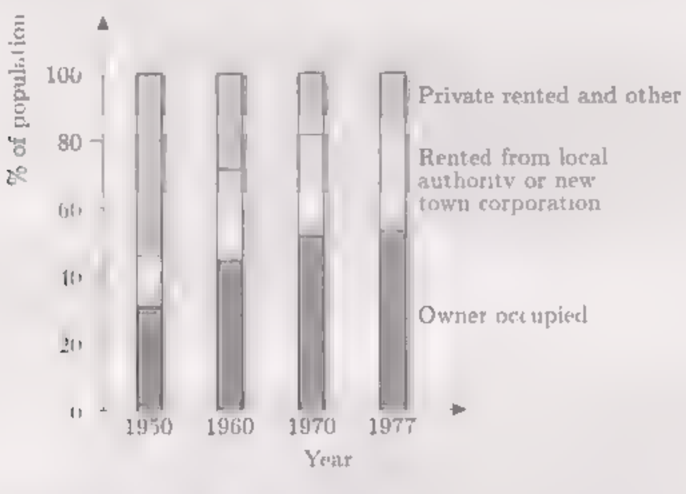


These histograms show that in 1968 the most frequently reported time was between 30 and 34 minutes, but in 1972 it was between 40 and 44 minutes. So the journey times seem to be getting longer. But it is not particularly easy to compare the information by looking at the histograms separately; it is much clearer if the histograms are combined onto the same diagram.



This **comparative chart** illustrates the differences between 1968 and 1972. The dark bars appear to be clustered to the left, whereas the light bars appear to be clustered to the right. This indicates that the journey times tended, overall, to be shorter in 1968 than they were in 1972. A comparative chart allows quick comparison of several sets of similar data.

Bar charts can also be stacked on top of each other to compare proportions. The bar chart below indicates the proportions of different types of living accommodation in the UK between 1950 and 1977.

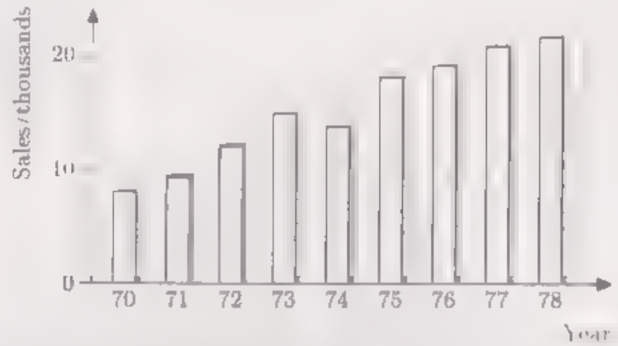


This bar chart shows that the proportion of owner occupiers increased from about 30% in 1950 to about 50% in 1977. The proportion of accommodation rented from the local authority also increased, from about 15% to about 35%. The proportion of privately rented and other accommodation decreased significantly from nearly 60% to about 15%. (This does not necessarily mean that fewer people lived in privately rented accommodation in 1977, just that a smaller *proportion* of the total population were thus housed. The population of the UK has increased substantially since 1950, so, although the percentage dropped dramatically, the actual numbers may have stayed about the same, or if they did drop the decrease may have been considerably less than this bar chart suggests.)

Solutions on page 86.

Try some yourself (5.2.3)

1 The bar chart below indicates the numbers of cars sold by one firm in successive years.



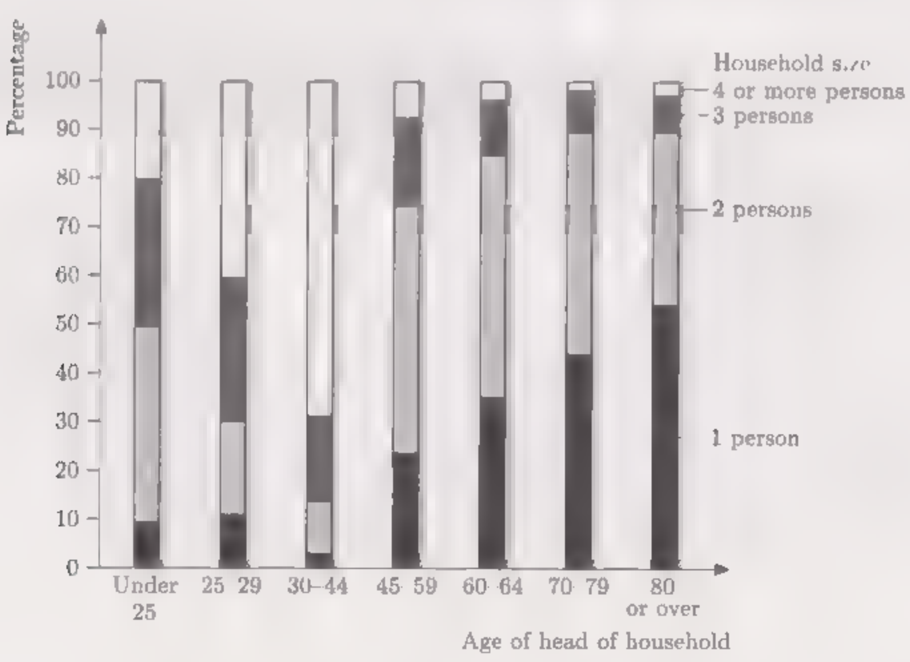
(a) How many cars were sold in 1972?
(b) How many cars were sold in 1977?
(c) What does the bar chart suggest about the pattern of sales?

2 The following table summarizes the results of a survey of 300 people who were asked 'How long did it take you to get from Central London to Heathrow Airport?'.

Time (mins)	Number of people
20–24	30
25–29	60
30–34	110
35–39	50
40–44	30
45–49	10
50–54	10
Total	300

(a) Illustrate the information in a histogram. (You may find it easier to draw your histogram on graph paper.)
(b) What is the width of each interval?

3 The bar chart below compares household size with the age of the head of household.



(a) What percentage of under 25 head of households live on their own?

(b) What percentage of over 80 head of households live in a household of four or more people?

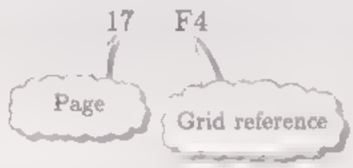
5.3 Graphs

5.3.1 Coordinates and scales

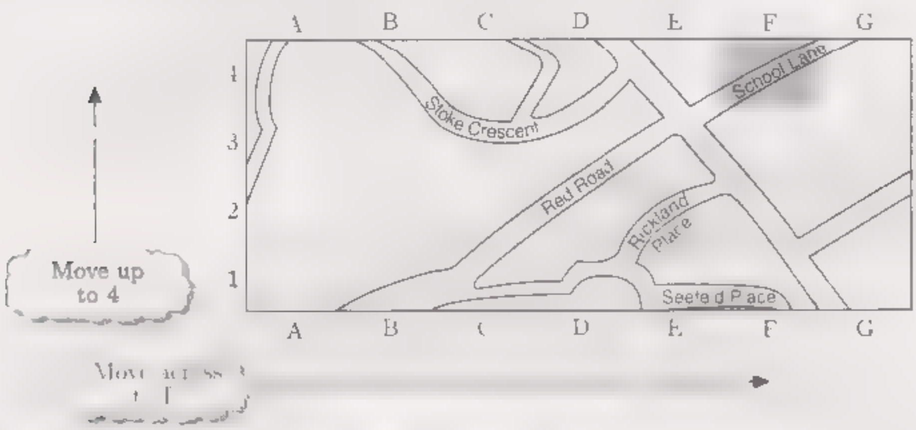
Many towns and cities have a book of street maps called an A to Z. You can look up the name of a street in the index and it will give you the page number of the map containing the street, plus the *grid reference* square for the street. There are different conventions for these grid references. You may have met several.

Example 7

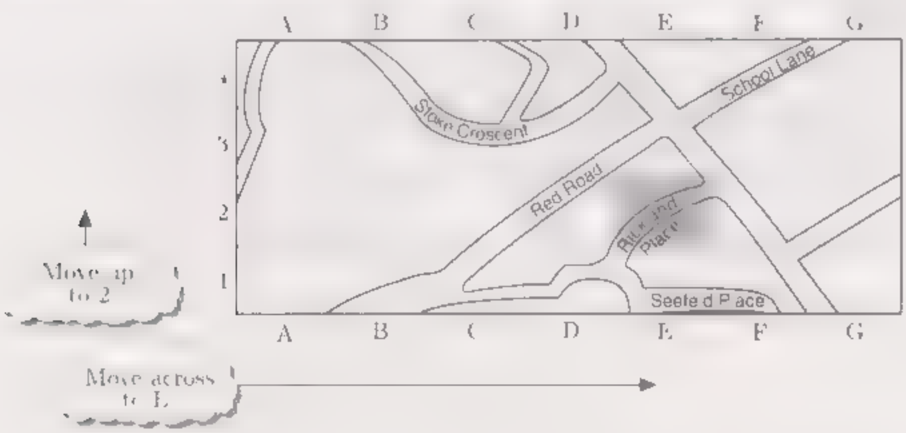
The index in an A to Z gives the reference for School Lane as:



To find School Lane on the map, turn to page 17, move across to column F and then up to row 4:



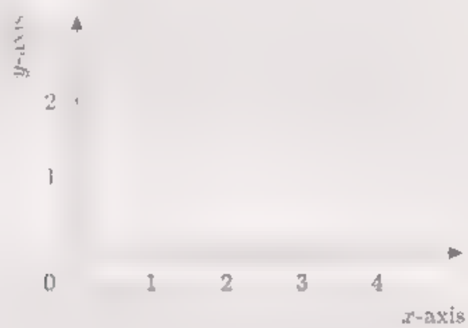
Similarly the grid reference for Rickland Place on page 17 is E2



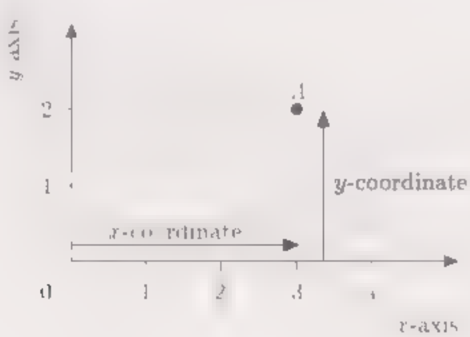
The procedure used to find grid references on a map is similar to the system used to locate points on a graph. For example, on the graph below the point A is located by moving across 3 and moving up 2.

In the same way that the grid references on a map are based on a starting point—in Example 7 the starting point is the bottom left corner (column A, row 1)—a starting point is needed to locate points on a graph. On a graph, the starting point is called the **origin**. From the origin you can move horizontally across and vertically up: the line across is called the **horizontal axis**; the up line is called the **vertical axis**.

On mathematical graphs, the horizontal axis is often labelled the ***x*-axis**, and the vertical axis is often labelled the ***y*-axis**. Scales are indicated on the axes to aid the location of points, and the origin is usually labelled 0.



The distance *across* to a point is called the ***x*-coordinate** of the point and the distance *up* to a point is called the ***y*-coordinate** of the point. So, in our example, A is located at the point with x-coordinate 3 and y-coordinate 2.

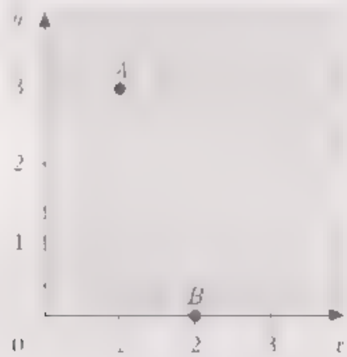


The coordinates of a point are written in brackets, x-coordinate followed by y-coordinate, separated by a comma. So for our point A we have:

$$(\underbrace{3}_{x\text{-coordinate}}, \underbrace{2}_{y\text{-coordinate}})$$

Example 8

Write down the coordinates of the points A and B shown below.



Solution

To locate A, we:

> move *across* 1 unit;
> move *up* 3 units.

The coordinates of A are $(1, 3)$.

To locate B, we:

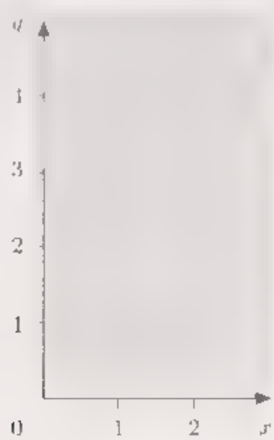
> move *across* 2 units;
> move *up* 0 units.

The coordinates of B are $(2, 0)$.

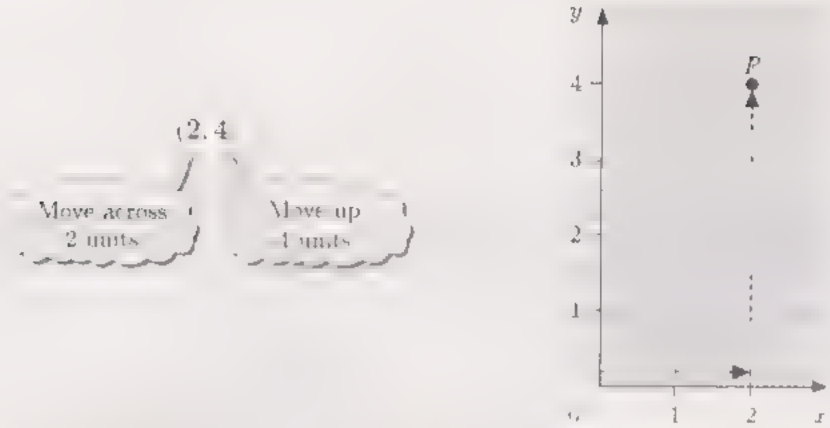
The process used to locate points can also be used to plot points.

Example 9

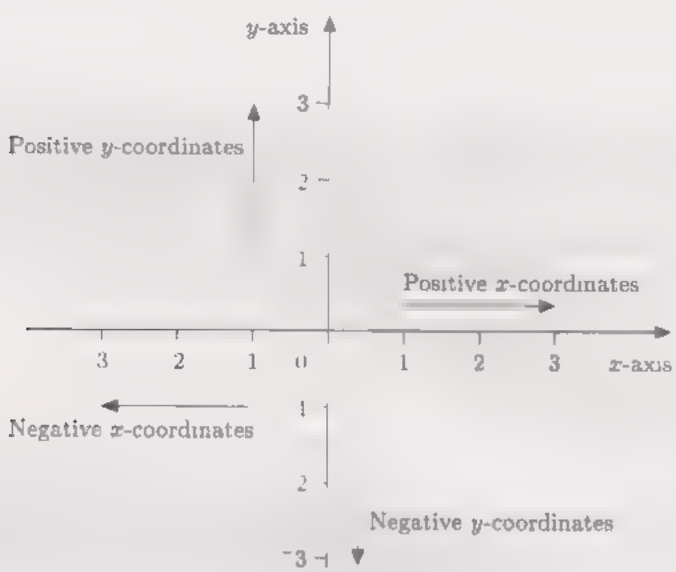
Plot the point P with coordinates $(2, 4)$ on the diagram below.



Solution



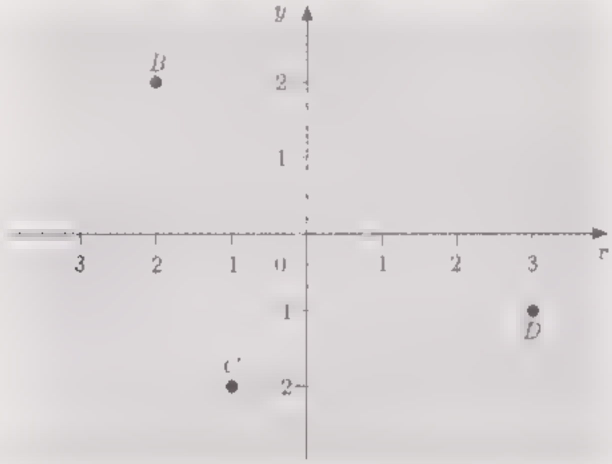
Up till now we have considered only those points with positive or zero coordinates, points like (3, 4), (4, 7) or (0, 1). However, the system can be extended to cope with points involving negative coordinates, such as ($^-2$, 3) or ($^-2$, $^-3$). Just as the number line is extended to cope with negative numbers, the x-axis and y-axis can be extended to cope with negative coordinates.



In this way, if a point is to the left of the origin, its x-coordinate is negative and if it is below the origin, its y-coordinate is negative.

Example 10

Write down the coordinates of the points B, C and D shown below.



Solution

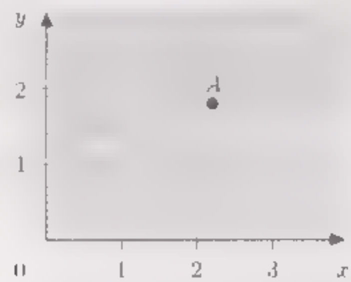
The point B is located 2 units to the left of the origin and 2 units up from the origin. So the coordinates of B are $(^-2, 2)$.

The point C is located 1 unit to the left of the origin and 2 units below it. So the coordinates of C are $(^-1, ^-2)$.

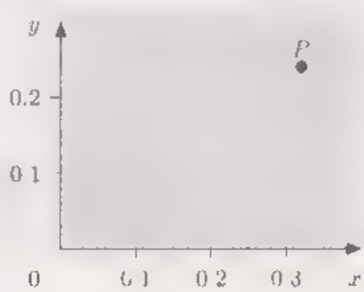
The point D is located 3 units to the right of the origin and 1 unit below it. So the coordinates of D are $(3, ^-1)$.

Remember the x-coordinate is given first, the y-coordinate is given second.

The coordinates of points can also be given using decimals or fractions. Locating points whose coordinates are not whole numbers requires more precision when measuring along the axes. For example, here the coordinates of the point A are $(2.2, 1.8)$.

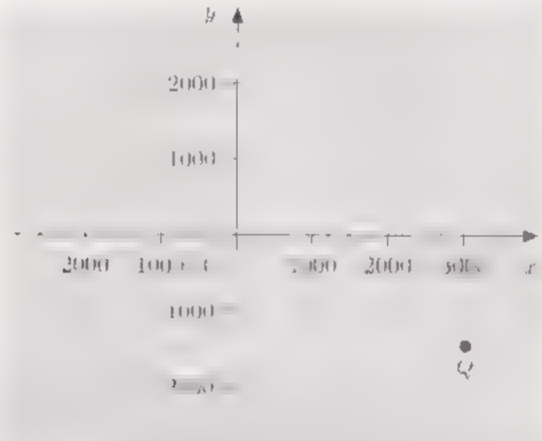


However, the point P with coordinates $(0.32, 0.24)$ would be difficult to plot on the axes above because of the scales used. We can only plot the point approximately. But, if a larger scale is used for the axes, it becomes quite easy to plot P reasonably accurately.

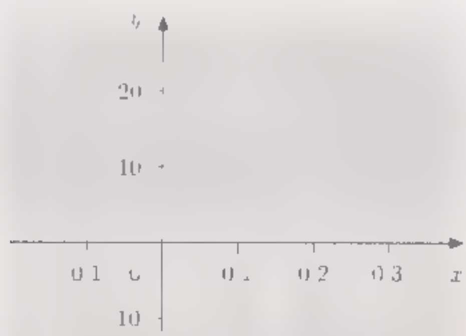


Similarly, to plot points with very large coordinates, such as the point Q with coordinates $(3020, ^{-}1450)$, a smaller scale is needed.

Notice that, even using scales like these, it's impossible to plot points such as Q exactly.



The scales on the x-axis and the y-axis do not necessarily have to be the same. Below the scale on the x-axis is 1 cm to 0.1 unit and the scale on the y-axis is 1 cm to 10 units.

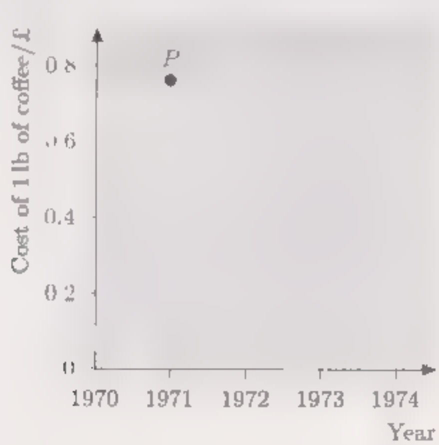


Because scales play such an important part in the location of coordinates, it's essential that they should be clearly indicated on the axes. Also, the axes should indicate exactly what is being measured and the units of measurement used.

Example 11

Write down the coordinates of the point P on the diagram below, and interpret those coordinates in terms of the labelling of the axes.

Notice that the horizontal axis starts at 1970, rather than 0.



Solution

In this example the horizontal axis measures years and the vertical axis measures the cost of 1 lb of coffee in pounds (£). The coordinates of P are $(1971, 0.76)$. This indicates that in 1971 the cost of 1 lb of coffee was £0.76 (or 76 p).

Usually, if you've performed an experiment or gathered some information, you will find that *you* have to decide what the axes should measure and what scales to use. This is probably the hardest part of plotting points and it's easy to go wrong at first. Ideally the po.nts should be clear and should fill the available space sensibly.

Good Bad

Furthermore, because graph paper is usually marked off in squares of 10 units or 5 units, it makes sense to use scales such as:

1 large square: 1 unit

1 large square: 5 units

1 large square: 10 units

Sometimes, however, other choices of scale can be sensible too.

Example 12

Choose suitable axes and plot the points $(11, 42)$, $(15, 68)$, $(3, 59)$ and $(5, 72)$ on the graph paper below.

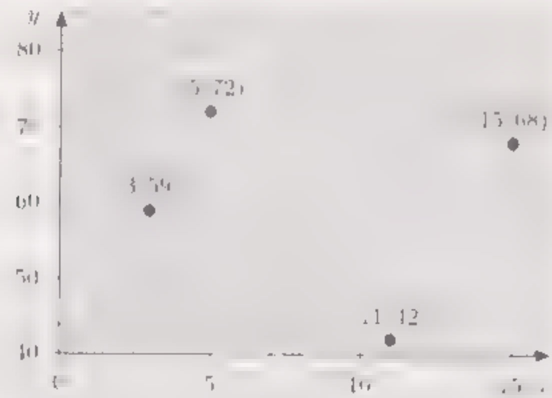
Solution

First look at the ranges of the coordinates.

The x-coordinates range from 3 to 15: the smallest is 3; the largest is 15. This suggests that the x-axis should range from 0 to 15. Comparing the range of coordinates with the size of the graph paper, a suitable scale for the x-axis is 2 large squares to 5 units.

The origin (0, 0) does not necessarily have to be included. As long as the scales are clearly indicated there will be no confusion.

The y-coordinates range from 42 to 72. We could start at 0 and use a scale of 5 small squares to 10 units. However, since all the y-coordinates are greater than 40, it is probably more practical to start the scale at 40 and use a scale of 1 large square to 10 units.



It is not easy to choose scales or to decide where the axes should cross. At first you will probably have to use trial and error. As you become more experienced you will find that it becomes easier. The important thing is to ensure that all the points fit within your axes. It's a good idea to use a pencil first—then it's easy to correct mistakes. You should be able to judge yourself whether the axes fit the paper well or whether they are squashed into one corner. Remember to consider whether it might be easier to choose convenient scales with the paper turned round, to have the shorter side at the bottom.

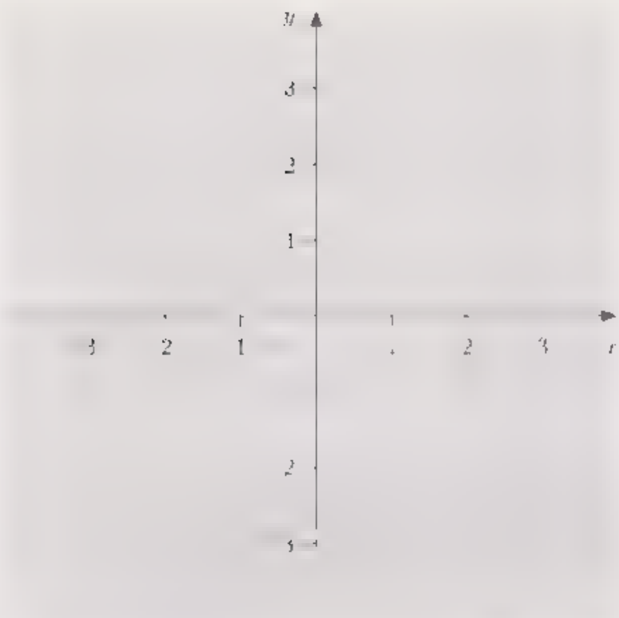
Solutions on page 87

Try some yourself (5.3.1)

1 Write down the coordinates of the following points A, B, C, D and E shown below.

2 Plot the following points on the diagram below:

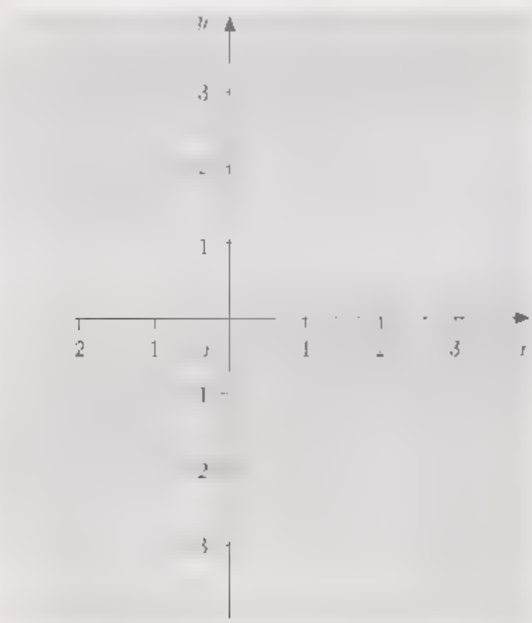
$P\,(2,3)$ $\quad Q\,(^-2,1)$ $\quad R\,(^-3,3)$ $\quad S\,(^-1,^-2)$ $\quad T\,(0,^-3)$ $\quad U\,(3,^-1)$



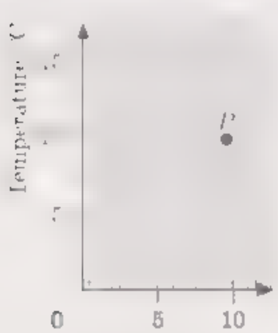
3 Plot the following points on the diagram below:

$A\ (2\frac{1}{2},$ [illegible]$)$ $B\ (0.8,\ 2.2)$ $C\ ($[illegible]$1.8,$ [illegible]$2)$

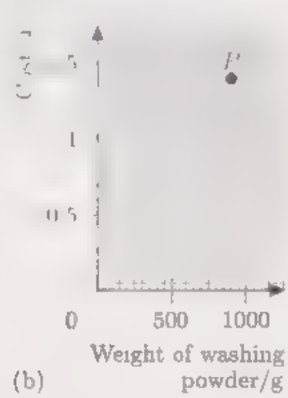
$D\ (^-1\frac{1}{2},\ ^-2\frac{1}{4})$ $E\ (1.3,\ 2.4)$ $F\ ($[illegible]$,\ \frac{2}{3})$



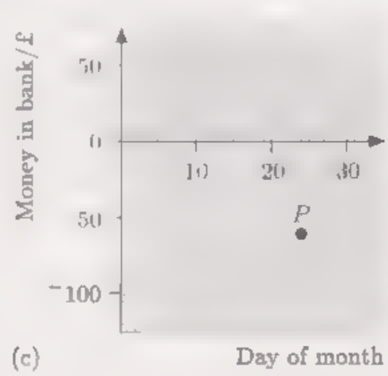
4 For each of the following, write down the coordinates of the point P and interpret those coordinates in terms of the labelling of the axes.



(a)



(b)



(c)

5 Plot each of the following sets of points within suitable axes on graph paper:

(a) $(2, 10)$, $(4, 38)$, $(7, 26)$, $(5, 23)$;

(b) $(350, 150)$, $(420, 168)$, $(630, 172)$, $(570, 159)$;

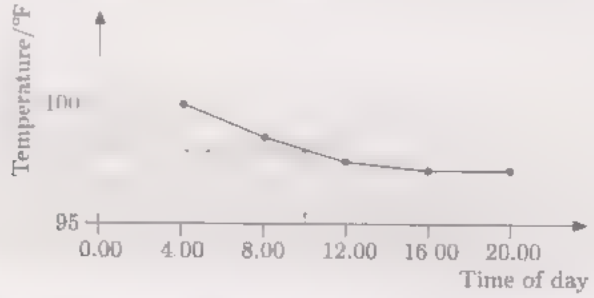
(c) $(140, 6)$, $(^{-}100, 30)$, $(60, 13)$, $(\ 60, 22)$.

5.3.2 Drawing and interpreting graphs

Graphs can be formed by joining up sets of plotted points. A graph provides more information than isolated points. It gives a better picture of a relationship and allows you to predict what happens between the known points.

For example, the temperature chart below indicates the hourly progress of a hospital patient. By joining up the plotted points we can see clearly how the patient's temperature dropped. And although his temperature was not taken at 10.00, the graph indicates that it was about 98 °F.

In this case we have joined up the points using straight lines. In other cases the points may be joined using smooth curves, as you will see below



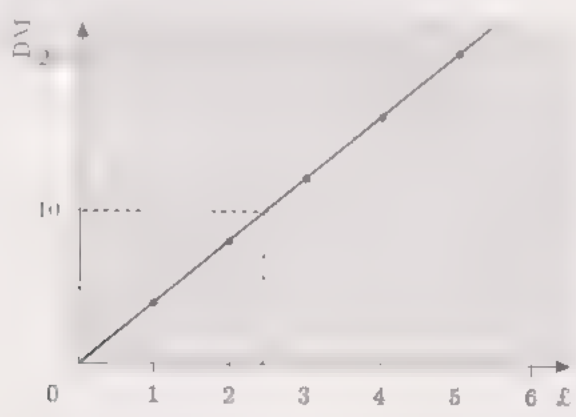
Example 13

The table below converts pounds sterling (£) to Deutschmarks (DM) based upon the exchange rate in July 1980.

£	1	2	3	4	5
DM	4.1	8.2	12 3	16.4	20.5

Plot a graph to illustrate this relationship. From the graph find the sterling equivalent of 10 DM.

Solution

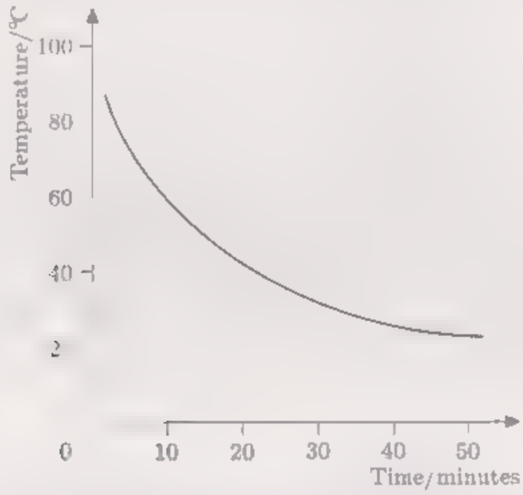


The points lie on a straight line. This line goes through the point at approximately $(2.45, 10)$ which means that 10 DM is equivalent to about £2.45.

Graphs often help us to visualize relationships better than tables of data. For example, the table below resulted from an experiment to investigate how quickly a cup of tea cooled down.

Time (mins)	0	5	10	15	20	25	30	35	40	45	50
Temperature (°C)	85	78	55	50	46	40	35	30	25	24	23

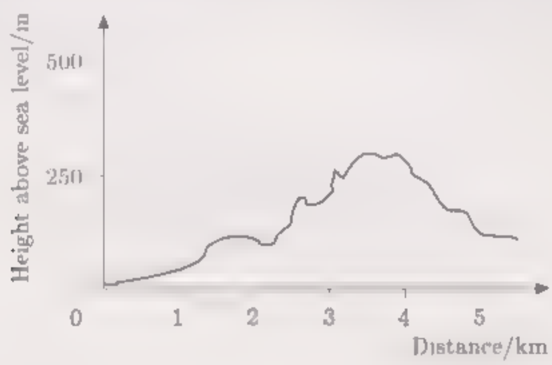
The resulting graph shows much more clearly than does the table how the temperature drops quickly at first, and then its rate of decrease slows down.



Temperature and time are the *variables* in this graph. We have indicated that time is measured along the horizontal axis and temperature is measured up the vertical axis. It is not always obvious which variable should be measured along which axis (though it is usual to measure time along the horizontal axis). It often doesn't matter: the important thing is that the axes are clearly labelled (and the units of measurement are clearly indicated).

Example 14

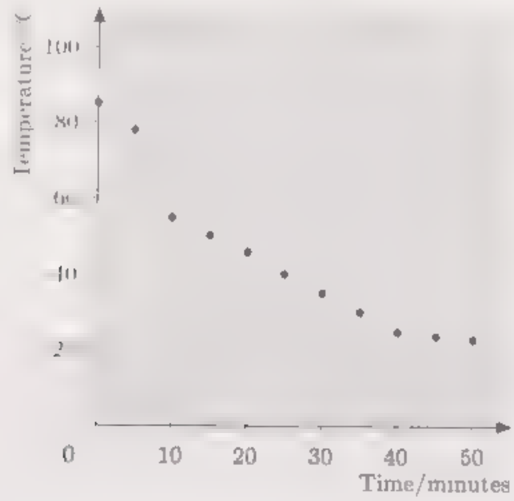
The diagram below shows the graph of height above sea level against distance from the coast. Interpret what the graph shows.



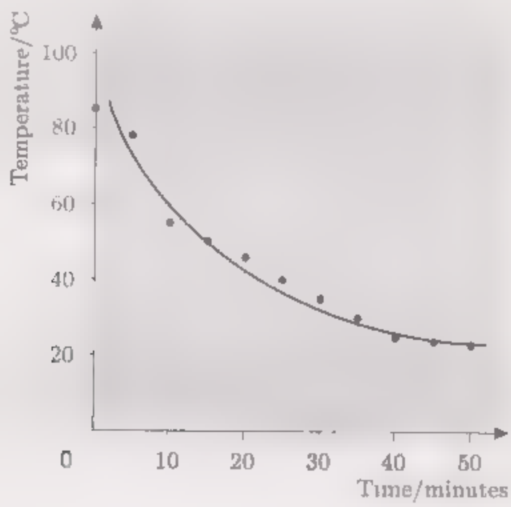
Solution

The graph shows that, starting at sea level, the height changes slowly at first so the terrain is quite flat; then it rises and increases irregularly after about 1 km, eventually reaching about 300 metres after about 3.5 km, before falling again.

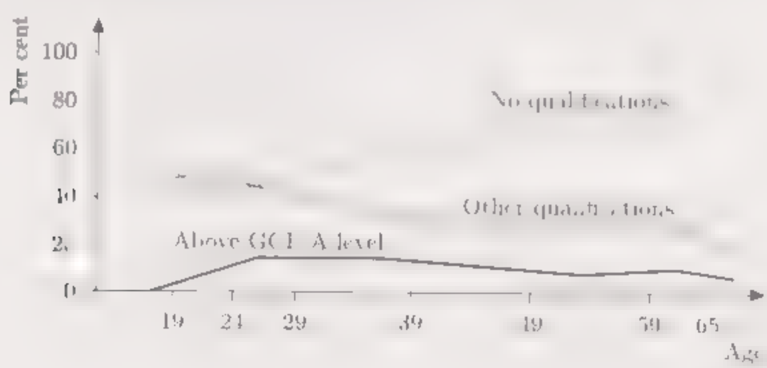
Usually, when a graph is plotted from a table of data, the points do not lie *exactly* on a curve. For example, the experiment on the cooling rate of tea actually produced the following points:



The points almost lie on a smooth curve—but not exactly. In such cases the graph is completed by drawing the smoothest curve possible. The graph illustrating the cooling rate of tea can therefore be completed as follows:



Sometimes several graphs are shown on the same diagram so that the reader can *compare* the information. For example, the diagram below shows graphs of the percentages of men in a certain community with various qualifications relative to age.



These graphs can be analysed separately. But it is also possible to compare the information from them. For example, the following features can be identified:

- ◇ A higher percentage of younger men (over 20) have some form of qualification compared with older men. (This perhaps reflects the availability of more, and more varied, educational opportunities at present compared with earlier times.)
- ◇ The percentage of men with qualifications above A-level remains fairly constant for men aged over 26.
- ◇ At the age of 49:

  about 10% have qualifications over A-level
  about 30% have some other qualification
  about 65% have no qualifications.

Notice that the total percentage is about $10 + 30 + 65 = 105\%$. The discrepancy arises because the percentages can only be read approximately from the graph

Try some yourself (5.3.2)

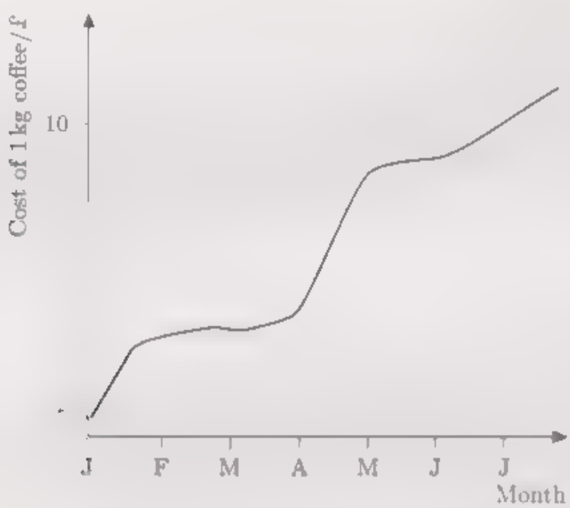
Solutions on page 88.

1 The table below converts pounds sterling (£) to dollars ($) based upon the exchange rate in July 1980.

£	5	10	15	20	25	30
$	11.5	23	34.5	46	57.5	69

Plot the points on graph paper. Complete the graph by joining up the points.

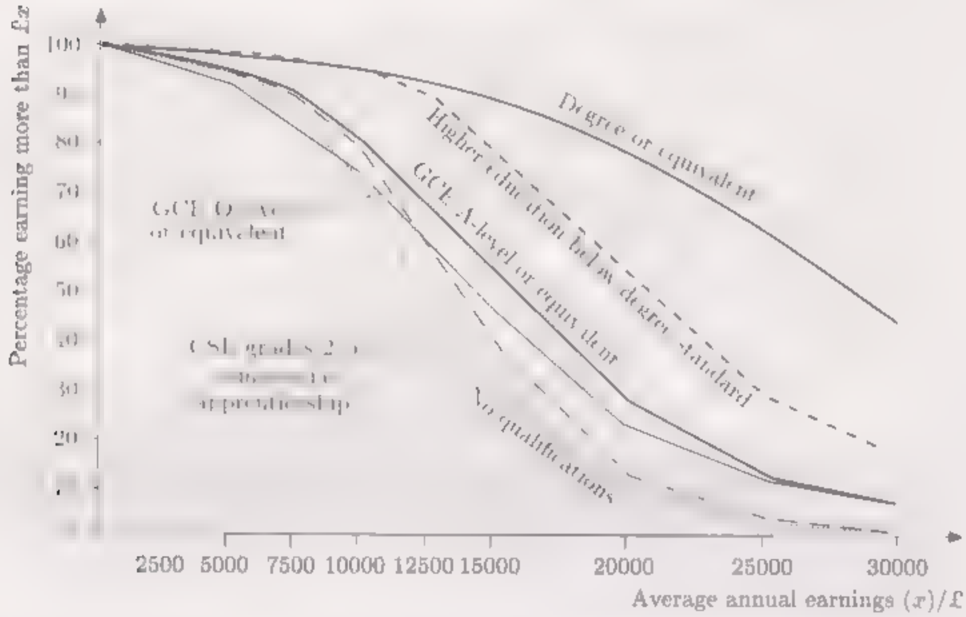
2 Summarize the information given by the graph below.



3 The table below gives the price of a particular car as it gets older. Plot a graph to illustrate the relationship.

Year	1982	1983	1984	1985	1986	1987	1988	1989
Price (£)	5000	3500	2500	1750	1150	900	650	500

4 The diagram below illustrates the relationship between annual income and qualifications for people in one area in one particular year.



(a) What percentage of people with a degree (or equivalent) earned more than £20 000 per annum?

(b) Summarize the information given by the graph representing 'no qualifications'.

(c) Compare the percentages in each category earning over £20 000 per annum.

(d) Compare the graphs showing 'degree or equivalent' and 'no qualifications'. What conclusions can you draw?

Outcomes

Now that you have studied Module 5 you should be able to:

- draw and interpret scale diagrams
- extract information from tables
- draw, interpret and compare pie charts, bar charts and histograms
- plot points and graphs by hand, using suitable axes and scales
- use and interpret coordinates;
- interpret and compare graphs

Module 6 Language, notation and formulas

Try these first

1 Read the following out aloud or write it out in full in words:

$$21 + 34 = 55.$$

2 Correct the grammar and punctuation of the following:

$$\frac{3^2 + 4^2}{5^2} \quad = 5^2 = 25 \quad = 4^2 = 16 \quad = 3^2 = 9 \quad = \frac{9 + 16}{25} = \frac{25}{25} = 1$$

3 Two labels have been missed off lines in the following. Where should they go to make sense of the argument?

$$10 + 12 - 20 = 2.$$

$$23 + 67 - 89 = 1.$$

So $(10 + 12 - 20) \times (23 + 67 - 89) = 2.$

$$32 + 23 - 55 = 0.$$

Hence using (1) and (2) means that

$$(10 + 12 - 20) \times (23 + 67 - 89) + 32 + 23 - 55 = 2 + 0 = 2.$$

4 Which mathematical operations are implied by the following words used in a mathematical sense?

(a) sum

(b) difference

(c) product

(d) quotient

State the operations in both words and symbols.

5 What do the following symbols mean?

(a) $=$

(b) $<$

(c) $>$

(d) $\Rightarrow$

6 Suppose you were selling a new lawn edging product and needed to know the perimeter of lawns to quote a price to prospective customers. The formula for the distance round the perimeter of a rectangular lawn is:

perimeter = 2 × length + 2 × breadth

Use the formula to find the perimeter of 6 metre by 8 metre lawn.

7 You sell this lawn edging product by the metre, but some customers have measured their lawns in feet and you have to convert their measurements to metres.

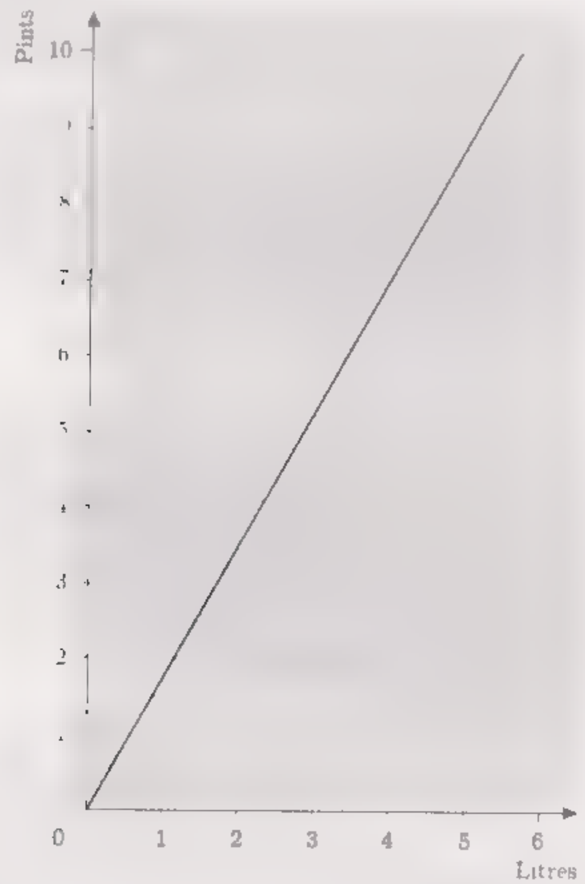
(a) You know that

$$1 \text{ foot} = 0.3048 \text{ metres}.$$

Use this to convert the measurement of a lawn 10 feet by 12.5 feet into metres.

(b) Use the formula given in Question 6 to find the perimeter of the lawn in metres, and then convert this back to feet.

8 Some products are sold by the pint and some by the litre. The conversion graph below has been devised to help customers convert quickly between these units. Use the graph to convert 4 litres to pints and 5 pints to litres.



Check your answers

Section 6.1.1 *1* Twenty-one plus thirty-four equals fifty-five.

Section 6.1.1 *2* $5^2 = 25, \quad 4^2 = 16, \quad \text{and} \quad 3^2 = 9.$

$$\text{So } \frac{3^2 + 4^2}{5^2} = \frac{9 + 16}{25} = \frac{25}{25} = 1$$

Section 6.1.1 *3* $10 + 12 - 20 = 2.$

$23 + 67 - 89 = 1.$

So $(10 + 12 - 20) \times (23 + 67 - 89) = 2.$ (1)

$32 + 23 - 55 = 0.$ (2)

Hence using (1) and (2) means that

$$(10 + 12 - 20) \times (23 + 67 - 89) + 32 + 23 - 55 = 2 + 0 = 2.$$

4 The mathematical operations are: — Section 6.1.2

(a) sum: addition $+$ (b) difference: subtraction $-$

(c) product: multiplication $\times$ (d) quotient: division $\div$

5 The meanings of the symbols are — Section 6.1.3

(a) $=$ is equal to *or* equals *or* which equals

(b) $<$ is less than *or* which is less than

(c) $>$ is greater than *or* which is greater than

(d) $\Rightarrow$ implies that *or* which implies that

6 Length is 6 m and breadth is 8 m. So the formula gives — Section 6.2.1

$$\begin{aligned}\text{perimeter} &= 2 \times \text{length} + 2 \times \text{breadth}\\ &= 2 \times 6\,\text{m} + 2 \times 8\,\text{m} = 12\,\text{m} + 16\,\text{m} = 28\,\text{m}.\end{aligned}$$

The perimeter is 28 metres.

7 (a) 1 ft = 0.3048 m. So — Section 6.2.2. Note that ft is an abbreviation for foot or feet.

$$\begin{aligned}10\,\text{ft} &= 10 \times 0.3048\,\text{m} = 3.048\,\text{m},\\ 12.5\,\text{ft} &= 12.5 \times 0.3048\,\text{m} = 3.810\,\text{m}.\end{aligned}$$

So the lawn is 3.048 m by 3.810 m.

(b) The formula for the perimeter gives

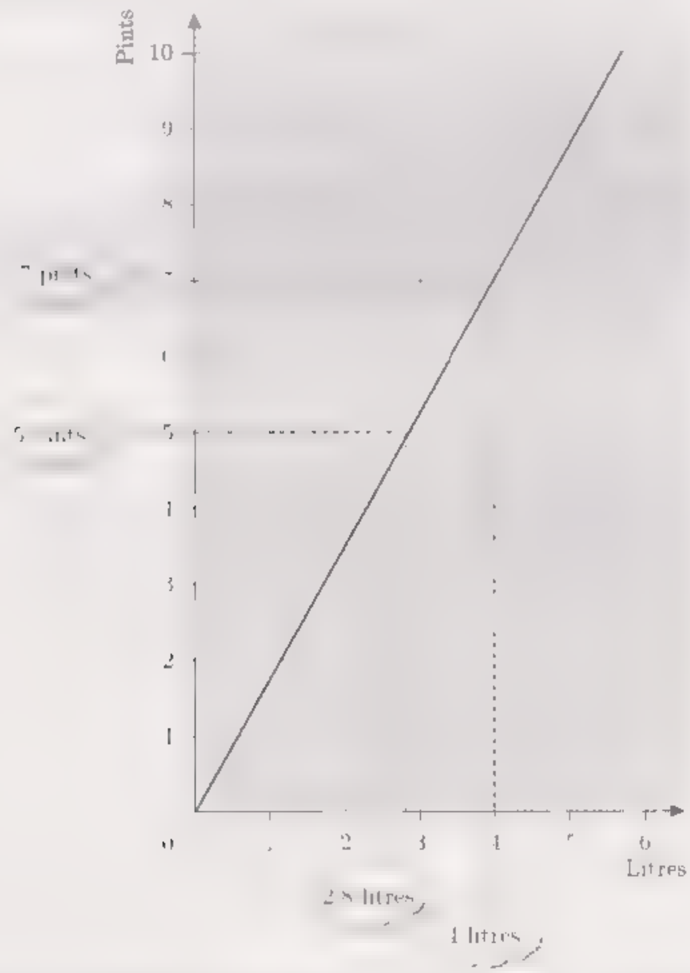
$$\begin{aligned}\text{perimeter} &= 2 \times \text{length} + 2 \times \text{breadth}\\ &= 2 \times 3.810 + 2 \times 3.048\,\text{m} = 13.716\,\text{m}.\end{aligned}$$

So the perimeter of the lawn in metres is 13.716 m.

0.3048 m = 1 ft, and so 1 m = 1/0.3048 ft. Therefore

$$13.716\,\text{m} = 13.716/0.3048\,\text{ft} = 45\,\text{ft}.$$

8 From the diagram below, 4 litres is 7 pints and 5 pints is 2.8 litres. — Section 6.2.3



6.1 Communicating mathematics

6.1.1 Layout

Talking or writing mathematics has a lot in common with talking or writing English. When you write mathematics, you should write in sentences, with full stops at the end. However, laying out mathematics differs from writing English in that in mathematics it is often best to start each sentence on a new line. Furthermore, each new line should follow on logically from the previous one.

When writing mathematics you can use a mixture of words and symbols—symbols in mathematics are just a shorthand for words. If you don't know the symbol, use the whole word. Make sure that you write so that somebody else can understand it—the somebody else might be yourself at a later date! It often helps to read aloud the mathematics that you have written to see if it make sense.

Example 1

Read the following mathematics aloud.

(a) $24.67 - 12.45 = 12.22.$ (b) $6.4 + 23.34 = 29.74.$

Solution

(a) Twenty-four point six seven minus twelve point four five is equal to twelve point two two.

(b) Six point four plus twenty-three point three four is equal to twenty-nine point seven four.

One of the most misused mathematical symbols is the equals sign =. It stands for the verb 'equals' or the phrases 'is equal to' or 'which is equal to' and so it should only come between two things which are equal.

Example 2

Which of the following equals signs should not be included in the following solution?

$$24.67 - 12.45 = 12.22.$$

$$6.4 + 23.34 = 29.74.$$

$$= \frac{24.67 - 12.45}{6.4 + 23.34} = \frac{12.22}{29.74}.$$

Solution

The one at the beginning of the third line. This equals signs is not between two things that are equal and so it should not be there. Also this is the beginning of a sentence, and when you read it aloud it does not make sense.

A lot of people use the equals sign wrongly in places where another word or phrase might actually help the meaning. For instance, at the beginning of a mathematical sentence, link words or phrases are sometimes useful. Examples include 'Thus', 'Hence', 'So', 'This implies' or 'It follows that'. Any of these would have been appropriate instead of the inappropriate

equals sign in the above example. Conjunctions such as 'and' and 'but' are also useful sometimes, as is punctuation, in particular full stops.

Example 3

Add useful words and punctuation to the calculation below to help somebody else follow it.

$$24.67 - 12.45 = 12.22$$
$$6.4 + 23.34 = 29.74$$
$$\frac{24.67 - 12.45}{6.4 + 23.34} = \frac{12.22}{29.74} = 0.410\,89 \text{ (5 dp)}$$

Solution

$$\begin{aligned} & 24.67 - 12.45 = 12.22 \\ \text{and} \quad & 6.4 + 23.34 = 29.74. \\ \text{So} \quad & \frac{24.67 - 12.45}{6.4 + 23.34} = \frac{12.22}{29.74} = 0.410\,89 \text{ (rounded to five decimal places).} \end{aligned}$$

Notice that mathematical calculations are often easier to read if all the equals signs are aligned underneath one another.

Sometimes you may want to refer back to sentences further up your work. You can label such sentences and then refer back to them by label. For example, an alternative way of laying out the above example might be:

$$24.67 - 12.45 = 12.22. \tag{1}$$
$$6.4 + 23.34 = 29.74. \tag{2}$$

So, using (1) and (2),

$$\frac{24.67 - 12.45}{6.4 + 23.34} = \frac{12.22}{29.74} = 0.410\,89 \text{ (rounded to five decimal places).}$$

Try some yourself (6.1.1)

Solutions on page 88

1 In the following two pieces of mathematical writing, remove or replace any inappropriate equals signs, and add in link words and punctuation to help somebody else understand.

(a) $2.3 + 3.7 = 6 \quad = 14.8 - 5.6 = 9.2$
$$= \frac{2.3 + 3.7}{14.8 - 5.6} = \frac{6}{9.2} = 0.65 \text{ (2 dp)}$$

(b) $(3.2)^2 = 10.24 \quad = (8.5)^2 = 72.25 \quad = 10.24 + 72.25 = 82.49$

2 Two labels have been omitted in the following. Where should they go to make sense of the argument?

$$2.3 + 3.7 = 6 \quad \text{and} \quad 14.8 - 5.6 = 9.2.$$

Hence $\dfrac{2.3 + 3.7}{14.8 - 5.6} = \dfrac{6}{9.2} = 0.652\,173\,913.$

Now $(3.2)^2 = 10.24$ and $(8.5)^2 = 72.25.$
So $(3.2)^2 + (8.5)^2 = 10.24 + 72.25 = 82.49.$
So, using (1) and (2),

$$\begin{aligned} \frac{2.3 + 3.7}{14.8 - 5.6} + (3.2)^2 + (8.5)^2 &= 0.652\,173\,913 + 82.49 \\ &= 83.14 \text{ (rounded to two decimal places).} \end{aligned}$$

6.1.2 Vocabulary

In mathematics some words are used in a more precise way than in English. It is important that a mathematical argument is unambiguous and so words which can be used in several contexts in English may only take one of these meanings mathematically. For instance, in English the word 'sum' might mean any calculation, but it has a precise mathematical meaning in the following: 'The sum of 456 and 789 is 1245'. Similarly, in English the word 'product' can mean a lot of things, but in mathematics the product of two numbers is the numbers multiplied together.

Many of the geometric terms are discussed in Module 7.

Here are a number of mathematical words and phrases, together with their mathematical meaning and examples of their use. Some of them you will have come across before; others may be new to you. Read them carefully and try to see the connection with other meanings which these words have in everyday English. You may well be able to add some of your own now or in your future studies.

Word	Mathematical meaning	Example of use
even	a whole number which is exactly divisible by two (i.e. which results in a whole number when divided by two)	432 is an even number
odd	a whole number which is not exactly divisible by two	54 321 is an odd number
sum	the addition of numbers	the sum of 123 and 456 is 579
difference	the numerical difference between two numbers or one number subtracted from the other	the difference between 342 and 324 is 18
product	the result of multiplying numbers together	the product of 3, 4 and 5 is 60
quotient	one number divided by another	the quotient $60 \div 12$ is equal to 5
exponent	the power to which a number is raised	the exponent of 2.6 is 4 in the expression $(2.6)^4$
perimeter	the outer boundary of a geometric figure	the length of the perimeter of the figure was 34 cm
circumference	the length of the perimeter of a geometric figure	the circumference of this circle is 34 mm.
centre	a point equidistant from the sides of a geometric figure	the centre of a circle is the same distance from all points on its perimeter
circle	a geometric figure for which every point on its perimeter is the same distance (the radius) from its centre	the circle has radius 4 cm
arc	part of the perimeter of a circle or other curved geometric figure	the length of the arc of a quarter circle is a quarter of the circle's circumference
right angle	the angle between two perpendicular lines	the angles at the corners of a square are right angles
parallelogram	a geometric figure with four straight sides and opposite sides of equal length and therefore parallel	a rectangle is a parallelogram with right angles at its corners
similar	the same shape	all squares are similar but all triangles are not
congruent	the same shape and the same size	congruent triangles have the same length sides and same size angles
equation	one expression equalling another	the left-hand side of an equation is always equal to the right-hand side
inequality	one expression being less than or greater than another	a negative number satisfies the inequality 'a number less than zero'

Try some yourself (6.1.2)

Solutions on page 89

1 Fill in the following table from your experience of earlier modules.

Word	Mathematical meaning	Example of use
decimal		
fraction		
positive		
negative		
scale		
triangle		
square		
rectangle		

6.1.3 Making sense of symbols

Mathematical symbols are a shorthand way of writing words or phrases that crop up quite often in mathematical writing. They are convenient abbreviations, some of which are in common usage by the general public and others which are less common.

For example, you have met the symbols +, −, × and ÷ before; but you may not have met some of the alternative ways of writing × and ÷

$+$ means 'plus' *or* 'add' *or* 'and'

$-$ means 'minus' *or* 'subtract' *or* 'take away'

Remember not to confuse the minus sign (−) with negative sign we use for negative numbers.

$\left.\begin{matrix}\times\\ *\end{matrix}\right\}$ both mean 'times' *or* 'multiplied by'

$\left.\begin{matrix}\div\\ /\end{matrix}\right\}$ both mean 'over' *or* 'divided by'

There are other alternatives to × and ∗ for multiplication. Sometimes a dot is used, provided there is no possible confusion with a decimal point. So, for example, $\frac{1}{2} \cdot \frac{1}{4}$ means 'a half times a quarter'. Sometimes, provided there is no risk of ambiguity, no symbol is used at all, just as sometimes we omit the words 'multiplied by' or 'times' in speech: we often say two fours or five fives, meaning two times four or five times five. So instead of writing $3 \times (4 + 5)$ we can just write $3(4 + 5)$ and save one symbol. Mathematicians like to be able to be as concise as possible!

Another alternative to ÷ and / for division is to use fractional notation. For example, $\frac{3+6}{4}$ means 'add three and six and then divide by four'. Furthermore, the / symbol is sometimes used for fractions, so that for example three-quarters is sometimes written as 3/4.

Other symbols you have met before include brackets, which are used to avoid ambiguity in mathematical expressions. You have also seen how we symbolize 'to the power': for example, 2^4 means '2 to the power 4'. An alternative to 2^4 is 2^4, where the symbol ^ means 'to the power'. A related symbol is the square root symbol.

The use of brackets was discussed in Module 1.

The ^ symbol is frequently used in computers.

$\sqrt{}$ means 'the square root of'

^ means 'to the power'

Example 4

What do the following expressions mean?

(a) $5/(3+2)$ (b) $5/3+2$ (c) $(5+8)\cdot 4$ (d) $(5+8)$^2 (e) $4(8-5)$

Solution

(a) 5 divided by the sum of 3 and 2 (giving 1).

(b) 5 divided by 3 and then added to 2 *or* $\frac{5}{3}$ added to 2 (giving $3\frac{2}{3}$).

(c) Add 5 and 8, and multiply the result by 4 (giving 52).

(d) Add 5 and 8, and square the result (giving 169).

(e) 4 times the difference between 8 and 5 (giving 12).

Two other commonly used symbols that you have met before are $=$ and $\simeq$. There are several others symbols of this type that you may meet during the course.

$=$ means 'equals' *or* 'is equal to' *or* 'which is equal to'
$\neq$ means 'does not equal' *or* 'is not equal to' *or* 'which is not equal to'
$\left.\begin{array}{l}\simeq\\ \doteq\end{array}\right\}$ both mean 'is approximately equal to' *or* 'which is approximately equal to'
$<$ means 'is less than' *or* 'which is less than'
$>$ means 'is greater than' *or* 'which is greater than'
$\leq$ means 'is less than or equal to' *or* 'which is less than or equal to'
$\geq$ means 'is greater than or equal to' *or* 'which is greater than or equal to'

Other symbols you have met before include the abbreviations for units of measurement: for example, m for metres, kg for kilograms and h for hours.

Example 5

What do the following mean?

(a) time > 2 s (b) height ≤ 100 m (c) $\frac{2}{3} \simeq 0.67$

Solution

(a) The time is greater than 2 seconds.

(b) The height is less than or equal to 100 metres.

(c) Two-thirds $(\frac{2}{3})$ is approximately equal to 0.67.

Other symbols you may come across and may find useful include the following.

$\equiv$ means 'is defined by'
$\Rightarrow$ means 'implies'
$\cdot$ means 'therefore'
means 'because'

You will meet many other symbols (and alternative meanings for some of those above) in your mathematical studies. It is important that you are sure what each one means in any given context, so that you can read and use it appropriately.

Try some yourself (6.1.3)

Solutions on page 89

1 What do the following mean?

(a) $(5+8)/(4-2)$ (b) $5+8/4-2$ (c) $(4+5)(5-2)$

(d) $9\sqrt{4}$ (e) $\frac{3}{4} \cdot \frac{5 * 6}{2}$ (f) $(\sqrt{25})$^3

2 What do the following mean?

(a) mass $\geq 10\,\text{kg}$ (b) time $< 2.4 \times 10^6\,\text{h}$ (c) $2/3 \neq 0.67$

6.2 Using formulas

A **formula** is a rule or a generalization, and as such *word formulas*—i.e. formulas using English words rather than mathematical symbols—are so much a part of life that people often use them without realizing that they are doing so. Here are some examples. The cost of tomatoes is the price per kilogram times the number of kilograms. The cost of petrol is the price per litre times the number of litres. The distance travelled is the average speed times the time taken. The number of tins of paint needed to cover a wall is the area of the wall divided by the area covered by one tin. All of these word formulas can be written more clearly using *some* mathematical symbols:

cost of tomatoes = (price per kilogram) × (number of kilograms)

cost of petrol = (price per litre) × (number of litres)

distance travelled = (average speed) × (time taken)

number of tins of paint = (area of wall)/(area covered by one tin)

6.2.1 Using a word formula

Word formulas are useful because you can use the same formula for a number of different calculations without having to work everything out from scratch each time. For instance, you can use the formula for the number of tins of paint for different rooms, the distance travelled formula can be used for different average speeds and journey times, and the formulas for the cost of petrol and tomatoes can used for different quantities and different prices.

Example 6

If the price of petrol is 59 pence a litre and a driver buys 25.9 litres, what is the cost?

Solution

The formula is

cost of petrol = (price per litre) × (number of litres).

The (price per litre) is 59 pence and the (number of litres) is 25.9. So the formula gives

cost of petrol = (price per litre) × (number of litres)

= 59 pence × 25.9 = 1528.1 pence.

Rounding to the nearest penny this is 1528 pence, or £15.28.

Example 7

Suppose you are travelling on a UK motorway in a coach and you wanted to know how far the coach had travelled since it joined the motorway two and a half hours previously. You glance over the shoulder of the driver and you see the speed is a steady 70 miles per hour. How far have you travelled on the motorway?

Solution

Using the formula

$$\text{distance travelled} = (\text{average speed}) \times (\text{time taken})$$

with (average speed) 70 miles per hour and (time taken) 2.5 hours gives:

$$\begin{aligned}\text{distance travelled} &= (\text{average speed}) \times (\text{time taken})\\ &= (70\text{ miles per hour}) \times (2.5\text{ hours})\\ &= 70 \times 2.5\text{ miles} = 175\text{ miles.}\end{aligned}$$

The units used in a formula must be consistent. For example, in the example above, since the average speed is in *miles* per *hour*, so the time travelled must be in *hours* and the distance travelled in *miles*. If one of the quantities is not in the correct units then you need to change it into the correct units before using the formula.

Example 8

Suppose you had been travelling at a speed of about 50 miles per hour for 45 minutes. How far would you have travelled?

Solution

You need to change the time into hours in order for the units to be consistent. 45 minutes is $\frac{3}{4}$ hour. So using the formula gives:

$$\begin{aligned}\text{distance travelled} &= (\text{average speed}) \times (\text{time taken})\\ &= 50 \times \tfrac{3}{4}\text{ miles} = 37.5\text{ miles.}\end{aligned}$$

Solutions on page 89.

Try some yourself (6.2.1)

1 If tomatoes cost 75 pence per kilogram, how much would 1.45 kilograms cost?

2 If you travel at a speed of about 60 kilometres per hour, how far would you have travelled after 1.5 hours, after 2 hours 40 minutes and after three and a half hours?

3 If you were going to paint rooms with wall areas of 56, 38 and 40 square metres with paint which came in tins which claimed to cover 15 square metres per tin, how many tins would you need?

6.2.2 *Converting units from a formula*

It is sometimes necessary to convert from one unit of measurement to another. You can use a formula in order to do this.

Example 9

1 pint = 0.5679 litres. How many litres is 4 pints?

Solution

If 1 pint is 0.5679 litres, then 4 pints will be 4 times this. So

$$4 \text{ pints} = 4 \times 0.5679 \text{ litres} = 2.2716 \text{ litres}.$$

Sometimes the conversion formula which you have is in the opposite direction from the one you want. However, it is possible to turn such formulas round. For example, since 1 km = 1000 m, you know that $1\,\text{m} = \frac{1}{1000}\,\text{km} = 0.001\,\text{km}$.

Example 10

1 mile = 1.6093 km. Turn the formula round to give 1 km in miles.

Solution

The formula can be written as

$$1.6093\,\text{km} = 1\,\text{mile}.$$

Hence, dividing by 1.6093 gives

$$1\,\text{km} = 1/1.6093\,\text{miles} = 0.6214\,\text{miles (rounded to four decimal places)}.$$

Try some yourself (6.2.2)

Solutions on page 90.

1 1 mile = 1.6093 km. What is 12 miles in kilometres?

2 Turn the following formulas round to give centimetres in terms of inches and grams in terms of pounds:

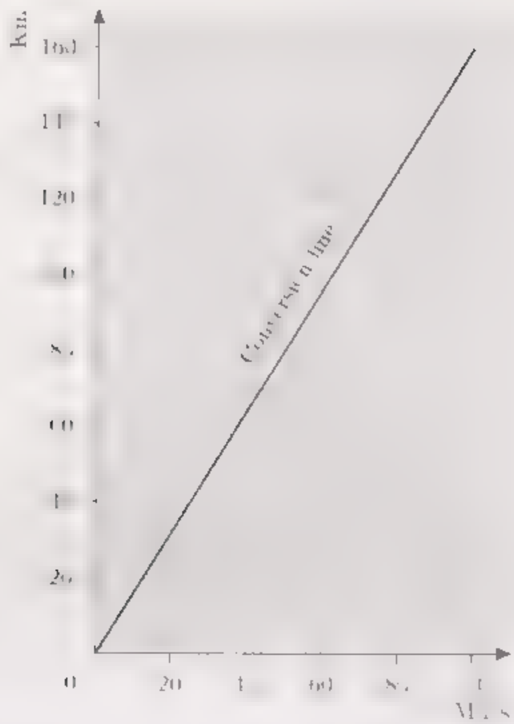
$$1 \text{ inch} = 2.54\,\text{cm}$$
$$1\,\text{lb} = 453.59\,\text{g}$$

How many pounds are there to the kilogram?

6.2.3 *Converting units from a graph*

If there are a lot of conversions to do between two sets of units, it is sometimes quicker to use a graph, though is unlikely to be as accurate as using a formula.

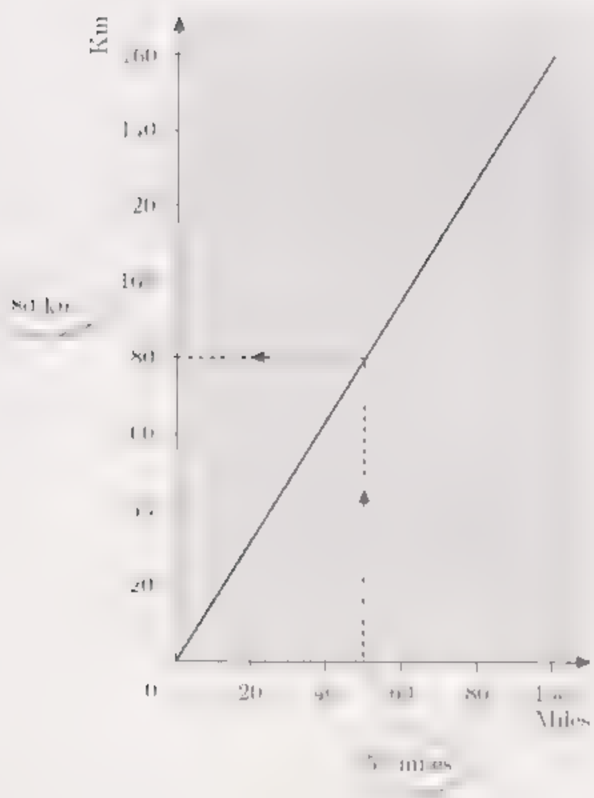
The graph below is a conversion graph from miles to kilometres. It would be useful for somebody planning a trip who wanted to convert a large number of distances from kilometres to miles (or vice versa).

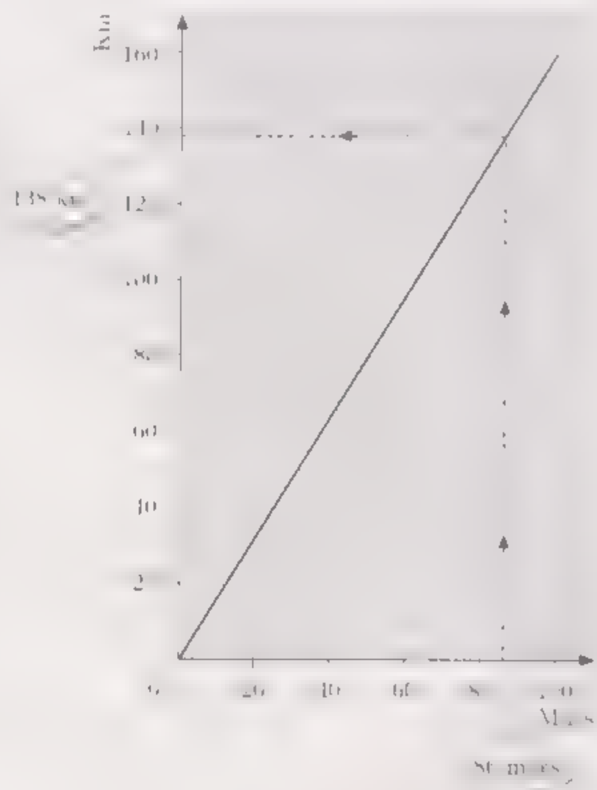


Example 11

Suppose you have a friend who is visiting the UK but who is used to distances measured in kilometres not miles, and you want to convert the distances between places from miles to kilometres for him. What are the following distances in kilometres: 50 miles, 86 miles?

Solution





Look at the conversion graph and locate 50 miles on the bottom (horizontal) axis. Go up to the conversion line and then along to the side (vertical) axis to give you 80 km, as shown in the left-hand diagram. So 50 miles is equivalent to 80 kilometres.

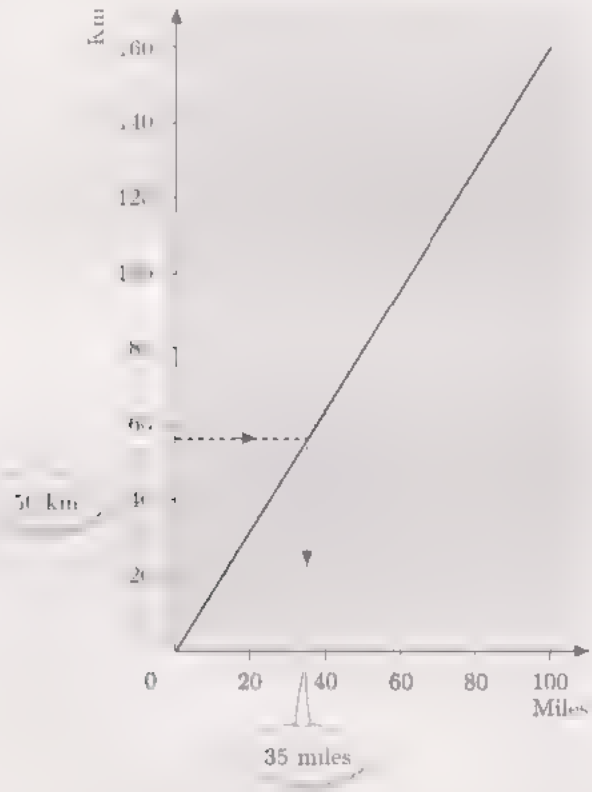
Now repeat this process for 86 miles, as shown in the right-hand diagram, and get 138 km.

The conversion graph can also be used to convert kilometres to miles, by the reverse process: i.e. locating the distance measured in kilometres on the vertical axis, going across to the conversion line and then down to the corresponding distance measured in miles on the horizontal axis.

Example 12

Convert 56 km to miles.

Solution



Locate 56 km on the vertical axis, go across to the conversion line, then down to the horizontal axis and read off 35 miles. So 56 km is 35 miles.

Try some yourself (6.2.3)

Solutions on page 90

1 Use the conversion graph to convert 25 miles and 96 miles to kilometres.

2 Use the conversion graph to convert 160 km, 120 km and 33 km to miles.

Outcomes

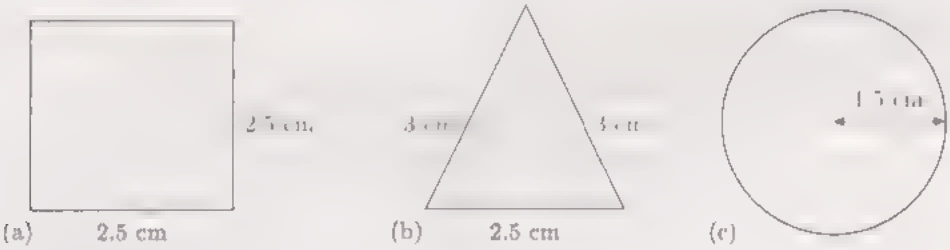
Now that you have studied Module 6 you should be able to:

- lay out, and label where appropriate, simple mathematical arguments;
- understand the precise mathematical meaning of certain common English words;
- understand and use common mathematical symbols;
- use word formulas;
- use formulas and graphs to convert units.

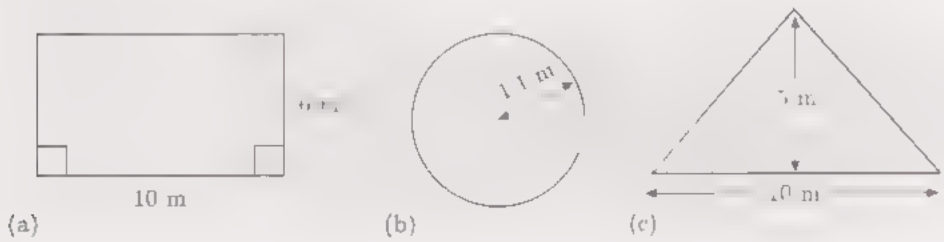
Module 7 Geometry

Try these first

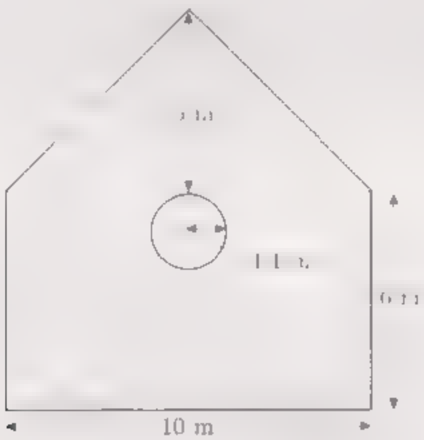
1 Describe the symmetry of the shapes below



2 Find the areas of the following shapes.

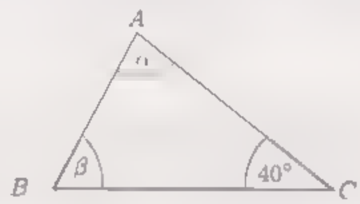


3 Use your answer to the previous question to find the area of turf needed for the proposed lawn shown below, with a circular flower bed in the middle. Round your answer to the nearest square metre.



4 Find the volume of water which the guttering shown below will hold, assuming that its cross section is semicircular.

5 This question concerns the following triangle:



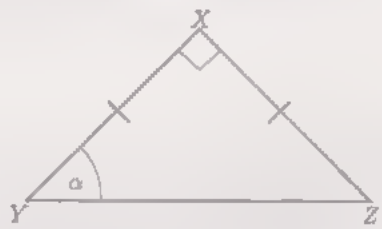
(a) Relabel α using an alternative notation.

(b) Measure α using a protractor.

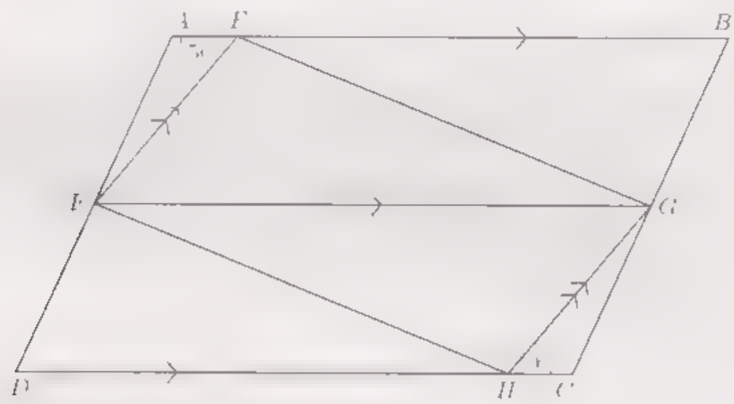
(c) What type of angle is α?

(d) Find β without using a protractor.

6 Deduce the value of α in the following triangle.

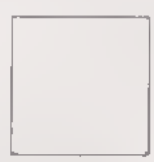


7 Find the value of α in the following diagram without using a protractor.



Check your answers

Section 7.1.1 *1* (a) This is a square, and so has four axes of reflectional symmetry (see below). It also has four-fold rotational symmetry about its centre.



square

(b) This is an isosceles triangle, and so has one line of reflectional symmetry (see below). It has no rotational symmetry.

isosceles triangle

(c) This is a circle, and so has an infinite number of lines of reflectional symmetry—every diameter is a line of reflectional symmetry. It also has complete rotational symmetry about its centre.

circle

2 (a) $10 \times 6 = 60\,\text{m}^2$ — Sections 7.1.2 and 7.1.3

(b) $\pi(1.1)^2 \simeq 3.80\,\text{m}^2$

(c) $\frac{1}{2} \times 10 \times 5 = 25\,\text{m}^2$

3 $(60 + 25 - 3.80)\,\text{m}^2 \simeq 81\,\text{m}^2$ to the nearest square metre. — Sections 7.1.2 and 7.1.3

4 The cross-section is a semicircle of radius 0.05 m. — Section 7.1.4

Area of semicircular cross-section $= \frac{1}{2}\pi(0.05)^2\,\text{m}^2 \simeq 0.003\,927\,\text{m}^2$.

Volume of guttering $\simeq 12 \times 0.003\,927 \simeq 0.047\,12\,\text{m}^3$.

So the guttering will hold about $0.047\,\text{m}^3$ of water.

5 (a) $\alpha = \widehat{A} = B\widehat{A}C = \angle BAC$. — Section 7.2.1

(b) $\alpha = 80\,°$. — Section 7.2.1

(c) α is an acute angle, since it is less than $90\,°$. — Section 7.2.1

(d) $\widehat{A} + \widehat{B} + \widehat{C} = 180\,°$ by the angle sum property of triangles. Hence $80\,° + \beta + 40\,° = 180\,°$, giving $\beta = 60\,°$. — Sections 7.2.2 and 7.2.3

6 The triangle is isosceles, so the base angles $\widehat{Y}$ and $\widehat{Z}$ are equal. Therefore $\widehat{Y} = \widehat{Z} = \alpha$. Also $\widehat{X} = 90\,°$. Hence, by the angle sum property of triangles, $90\,° + 2\alpha = 180\,°$, giving $\alpha = 45\,°$. — Sections 7.2.2 and 7.2.3

7 Using the alternate angles property, we have: — Section 7.2.4

AB and EG are parallel, so $G\widehat{E}F = A\widehat{F}E = 50\,°$;

EF and HG are parallel, so $E\widehat{G}H = G\widehat{E}F = 50\,°$;

EG and DC are parallel, so $C\widehat{H}G = E\widehat{G}H = 50\,°$.

Therefore $\alpha = C\widehat{H}G = 50\,°$.

7.1 Shapes, areas and volumes

7.1.1 Symmetry

When giving a description of a picture, shape or pattern it is often useful to refer to symmetry. In fact symmetry is a mathematical property and you will find that it is very useful to be able to 'spot' whether or not a shape is symmetrical.

Look at the shapes below. Describe the symmetry of the shape on the left. How is it related to the shape on the right?

You might like to think of the symmetry of the shape on the left in two ways.

(a) Imagine a mirror placed along central line. The reflection in the mirror gives the other half of the shape.

(b) Fold the shape along the central line. Then one side lies exactly on top of the other and all you would see would be the half shown in the diagram on the right.

This type of symmetry is called **reflectional symmetry**.

An **isosceles triangle** is a triangle with two sides of equal length. These are marked | in the diagram.

Any isosceles triangle has a reflectional symmetry:

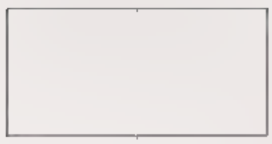
isosceles triangles

The following are all symmetrical:

The broken lines are called *lines of (reflectional) symmetry*, and each shape is said to be **symmetrical** about this line.

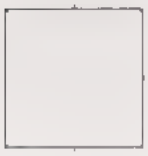
An **equilateral triangle** is a triangle with all three sides of equal length.

A shape can have more than one line of symmetry. For example, a rectangle has two lines of symmetry, an equilateral triangle has three and a square four.



rectangle

equilateral triangle



square

A circle has an infinite number of lines of symmetry since it can be folded about any diameter.

The **diameter** of a circle is a straight line through its centre stretching from one side of the circle to the other

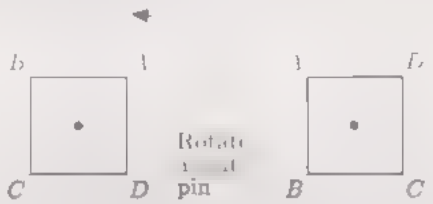
circle

Other shapes, such as a scalene triangle, have no lines of symmetry.

A scalene triangle is a triangle in which each side has a different length.

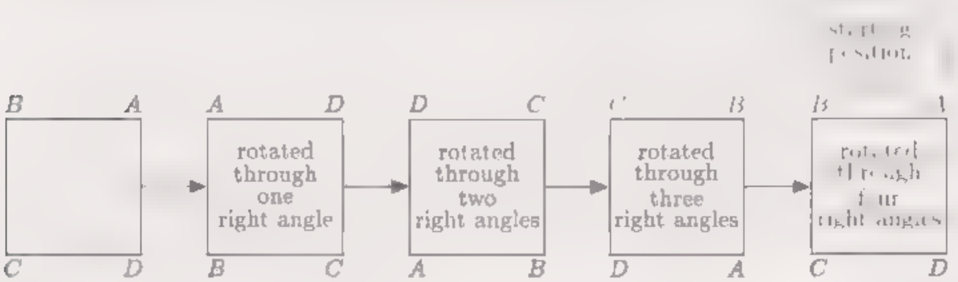
scalene triangle

A shape may also have **rotational symmetry**. Imagine a pin through the centre of a square. If the square is rotated through one right angle, the shape looks exactly the same.



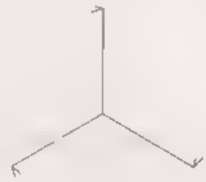
The corners of the square have been labelled to illustrate this property; the shape looks the same but the corners have been moved round one place. The square occupies exactly the same position. The centre point, which acts as a pivot, is called the **centre of rotation**.

The square looks the same if it is rotated through another right angle. Again the corners move around one place. If the process is repeated the square will eventually end up in the starting position with the corner A in the top right-hand corner. The diagram below illustrates what happens.



Four right angles comprise a complete revolution.

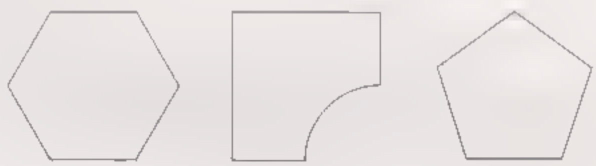
A square is said to have **rotational symmetry of order 4** or **four-fold rotational symmetry**, because after $\frac{1}{4}$ of a revolution (one right angle) the shape looks exactly the same and lies in the same position. The shape below has **rotational symmetry of order 3** or **three-fold rotational symmetry**, because after $\frac{1}{3}$ of a revolution the shape looks exactly the same and lies in the same position.



Solutions on page 91.

Try some yourself (7.1.1)

1 Mark the lines of symmetry on each of the following shapes. For each shape state the total number of lines of symmetry.



(a) (b) (c)

2 Mark the centre of rotation on each of the following shapes, and for each state the order of rotational symmetry.

(a) (b) (c) (d)

3 Describe the symmetry of each of the following shapes. In each case mark the lines of symmetry, and state their number, and mark the centre of rotation, and state the order of rotational symmetry.

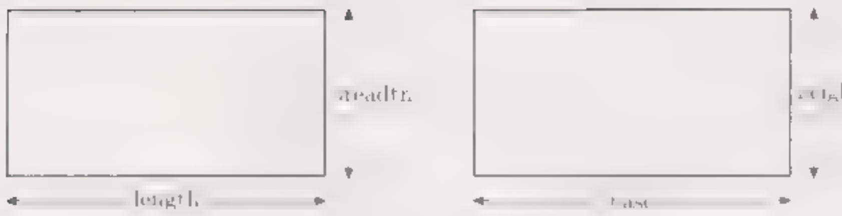
(a) (b) (c)

7.1.2 Areas of quadrilaterals and triangles

A **quadrilateral** is any shape with *four* straight sides and a **triangle** is any shape with *three* straight sides. We begin by looking at the areas of two types of quadrilateral.

A **rectangle** is a shape with four straight sides and right angles at its corners.

The area of a rectangle is its length times its breadth. However, sometimes the dimensions of a rectangle are referred to as base and height instead of length and breadth. In this case the area is the base times the height.



Area of rectangle = length × breadth = base × height

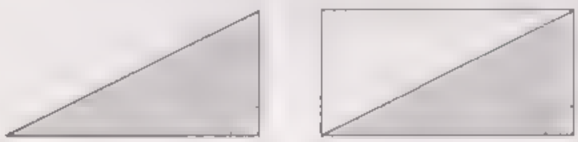
A square is a special kind of rectangle where the length is equal to the breadth. So its area is the length of its side times itself or the length of its side squared.

A **square** is a rectangle with four sides of the same length.

Area of square = (length of side) × (length of side) = (length of side)^2

Next we look at triangles. First consider a right-angled triangle, as it is easy to see that its area is exactly half of the area of a rectangle.

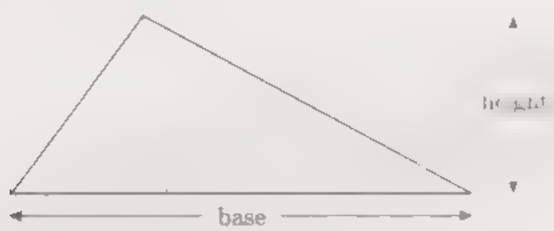
A **right-angled triangle** is a triangle in which one of the angles at its corners is a right angle.



Since the area of the rectangle is the base times the height, the area of the triangle is half of this.

Area of right-angled triangle = $\frac{1}{2}$ × base × height

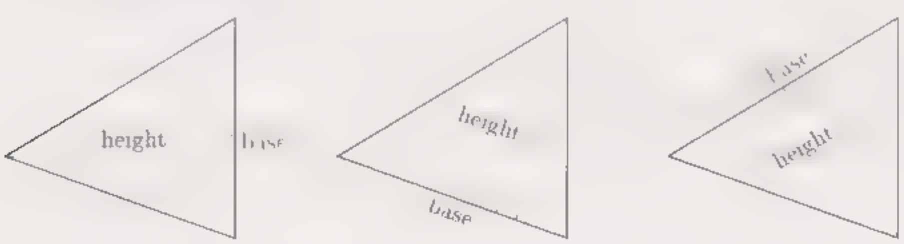
Not every triangle is right-angled, but the formula can be adapted to apply to *all* triangles since any triangle can be divided into two right-angled triangles, as shown below, and as such its area is half of the area of the corresponding rectangle.



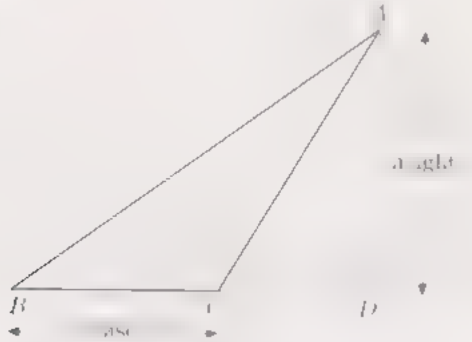
The height is the *perpendicular* distance from the base to the opposite vertex. In order to avoid ambiguity it is sometimes called the *perpendicular height* rather than just the height. So the general formula for the area of a triangle is as follows.

Area of triangle = $\frac{1}{2}$ × base × perpendicular height

It is not always easy to identify the 'base' of a triangle if it does not have a horizontal side, but the beauty of the formula is that it works no matter which side is called the base. Thus the area of the following triangle can be evaluated in three ways.



Sometimes the perpendicular height may lie outside the triangle, but the formula still holds.



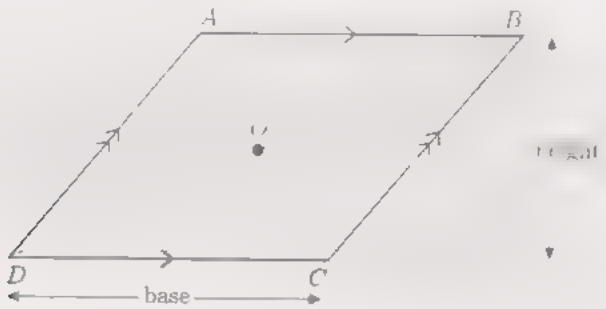
Don't worry if you can't follow this proof at the moment.

This can be proved as follows, where we have abbreviated the word triangle to the symbol Δ.

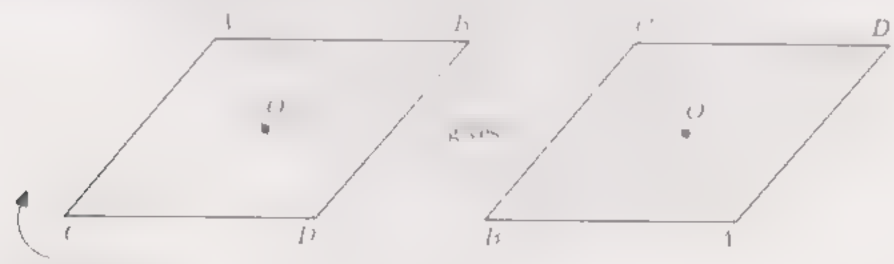
$$
\begin{aligned}
\text{Area } \Delta ABC &= \text{area } \Delta ABD - \text{area } \Delta ACD \\
&= \tfrac{1}{2} \times \text{height} \times BD - \tfrac{1}{2} \times \text{height} \times CD \\
&= \tfrac{1}{2} \times \text{height} \times (BD - CD) \\
&= \tfrac{1}{2} \times \text{height} \times BC \\
&= \tfrac{1}{2} \times \text{height} \times \text{base}
\end{aligned}
$$

You can use what you know about the areas of rectangles and triangles to find the areas of other quadrilaterals. For example, a **parallelogram** is a quadrilateral with opposite sides parallel. In fact opposite sides are also equal.

The arrows indicate that the corresponding sides are parallel.



Its area in the sum of the areas of two triangles: $\Delta ABD + \Delta CDB$. The area of $\Delta CDB = \frac{1}{2} \times$ base $\times$ height. If you rotate the parallelogram through half a revolution (two right angles) about the point O, you will see that ΔABD lies exactly over ΔCDB and so has the same area.



Hence we can deduce that:

Area $ABCD = 2 \times (\frac{1}{2} \times \text{base} \times \text{height}) = \text{base} \times \text{height}$

Therefore in general:

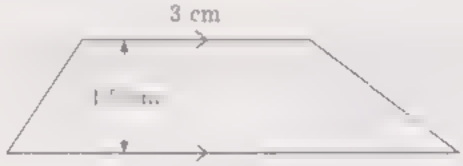
Area of parallelogram = base × perpendicular height

You can often split a complicated area into the sum of areas with simple shapes, or sometimes the area of one simple shape minus another.

Example 1

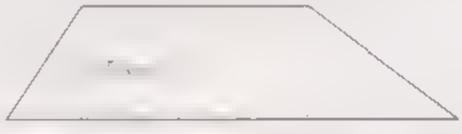
Find the area of the trapezium below.

A **trapezium** is a quadrilateral with two sides parallel.



Solution

We can divide the trapezium into three parts—two triangles and a rectangle—as shown.



Therefore we have:

Area of trapezium = area of left triangle + area of rectangle + area of right triangle

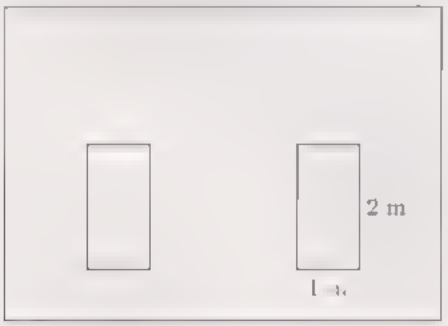
$$= (\tfrac{1}{2} \times 1 \times 1.5) + (3 \times 1.5) + (\tfrac{1}{2} \times 2 \times 1.5)\ \mathrm{cm}^2$$
$$= 0.75 + 4.5 + 1.5\,\mathrm{cm}^2$$
$$= 6.75\,\mathrm{cm}^2$$

Example 2

Suppose a friend of yours decides to lay crazy paving in his garden, which measures 7 m by 5 m, but he wants to leave two rectangular areas, each 2 m by 1 m, for flower beds. If it costs £8.50 to cover an area of $1\,\mathrm{m}^2$ with crazy paving, how much will it cost him to pave his garden?

Solution

This first thing that you should do with a problem like this is draw a diagram, and include on it all the information that you have been given.



From the diagram you have:

Area of the garden $= 7 \times 5\,\mathrm{m}^2 = 35\,\mathrm{m}^2$

Area of one flower bed $= 2 \times 1\,\mathrm{m}^2 = 2\,\mathrm{m}^2$

Therefore.

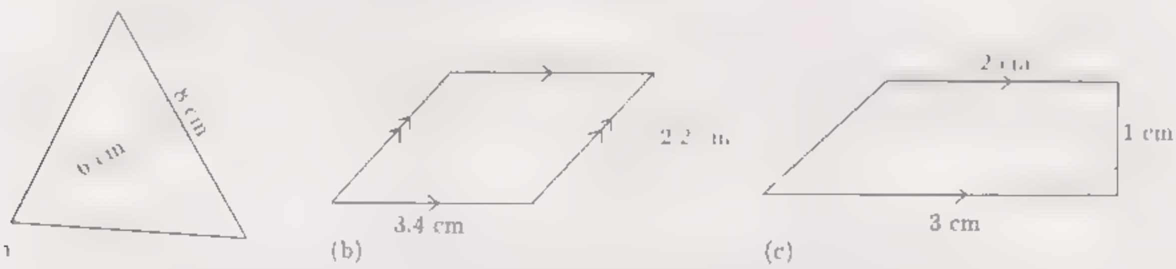
$$\text{Area of crazy paving} = \text{area of the garden} - (2 \times \text{area of one flower bed})$$
$$= 35 - (2 \times 2)\,\text{m}^2 = 35 - 4\,\text{m}^2 = 31\,\text{m}^2$$

So he needs to buy enough crazy paving to cover $31\,\text{m}^2$. This will cost $31 \times £8.50 = £263.50$.

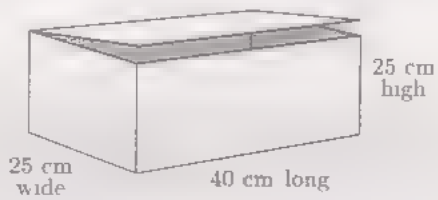
Solutions on page 91.

Try some yourself (7.1.2)

1 Find the areas of each of the shapes below

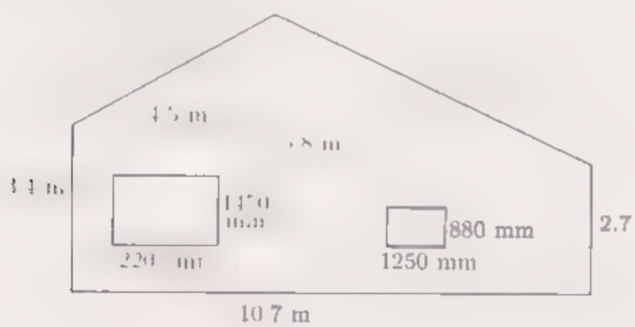


2 A girl has made a wooden sewing box and now wishes to line the inside with padded material. She wants to line all the sides, the base and the inside of the lid with the material, which she intends to cut into six pieces and stick on. Calculate what area of material she will need.

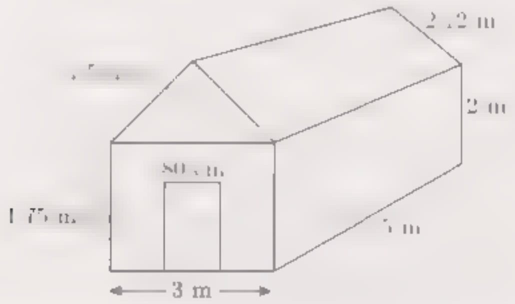


3 A carpet has an area of $6\,\text{m}^2$. It is to be laid onto floor that is 5 m long and 4 m wide. The floorboards not covered by the carpet are to be varnished. What area of floor will require to be varnished and how much varnish will be needed if a tin can cover $2.5\,\text{m}^2$?

4 The diagram shows the end wall of a bungalow, containing two windows. The wall is to be treated with a special protective paint. To decide how much paint he needs, the owner wants to know what the area of the wall is. Divide the wall up into simple shapes and hence find the total area.

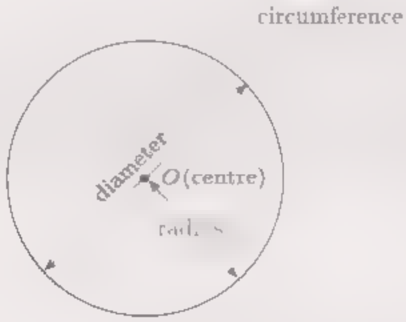


5 The diagram below shows the dimensions of a frame tent. Calculate the amount of canvas needed to make the tent if the door shown is made of net.



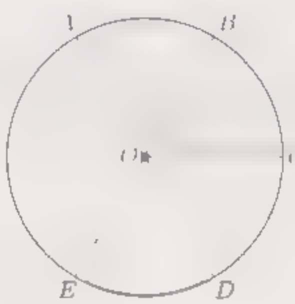
7.1.3 Circles

A **circle** is defined as the set of all points which are a fixed distance from a fixed point. This fixed point is called the **centre** of the circle and is often indicated with the letter O. The fixed distance is called the **radius** of the circle, twice this fixed distance is called the **diameter** of the circle and the distance around the circle, i.e. around its *perimeter*, is called the **circumference**.



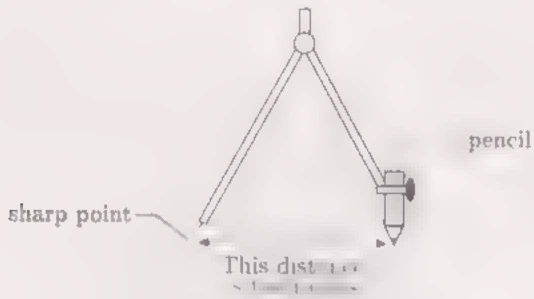
The words *radius*, *diameter* and *circumference* are also used to refer to lines as well as distances. The **circumference** of a circle is its perimeter line, a **radius** of a circle is a straight line joining the centre to a point on its circumference and a **diameter** of a circle is a straight line joining two points on the circumference and which passes through the centre of the circle. For example, in the circle below, OA, OB, OC, OD and OE are all radii and AD and BE are diameters. The points A, B, C, D and E all lie on the circumference.

The plural of radius is radii

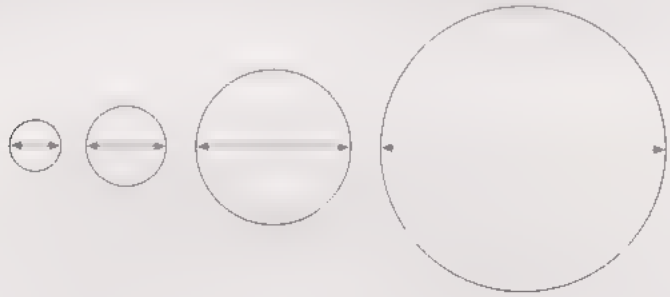


The best way to construct (or draw) a circle of a given radius is to use a pair of compasses. Using a ruler set the distance from the point of the compasses to the tip of the pencil to the desired radius. Place the point on

the paper at the position where you want the centre of the circle to be and carefully rotate the compasses on the point so that the pencil marks out the required circle.



The circles below indicate that as the diameter of a circle increases so does the circumference. What is less obvious is that there is a specific relationship between the circumference and diameter.



In fact the circumference is always a constant number times the diameter:

circumference = constant × diameter

π is the Greek letter for p

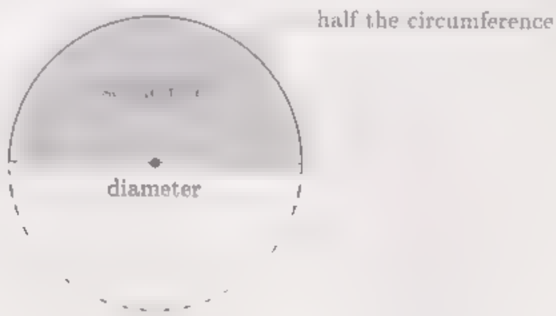
This constant number is given the name pi and is written π. Its value is approximately 3.14. So the formula for the circumference of a circle is:

circumference = π × diameter

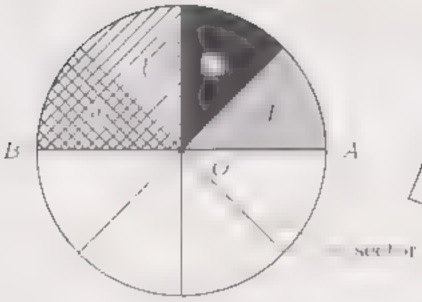
Since the diameter is twice the radius, this formula can also be written as:

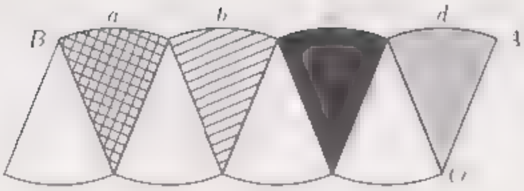
circumference = 2π × radius

Any diameter divides a circle into two equal parts, called **semicircles**.



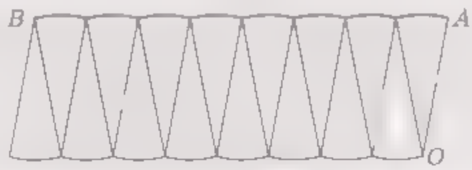
The circle below has been divided up into equal 'slices' or **sectors**, the shaded sectors being labelled a, b, c, d. The eight sectors can be cut out and rearranged into the shape shown, whose area is the same as that of the circle.





You can see that the top side AB of this shape has the length of half the circumference of the circle—i.e. $\frac{1}{2} \times 2\pi \times \text{radius} = \pi \times \text{radius}$. Also the length of the side OA is the same as the radius of the circle.

We can divide the circle into more and more sectors and rearrange them. For example, dividing the circle into 16 equal sectors gives the following shape, whose area again is the same as that of the circle.



Again the length of the top side AB is $\pi \times$ radius and the length of OA is the same as the radius

Notice how the rearranged shape is beginning to look more like a rectangle. The more sectors we divide the circle up into, the straighter AB will become and the more perpendicular OA will become. Eventually we will not be able to distinguish the rearranged shape from a rectangle. The area of this rectangle will be the same as that of the circle, and its sides will have length $\pi \times$ radius (for AB) and radius (for OA). So we can deduce the following formula:

$$\begin{aligned}\text{Area of circle} &= \text{area of rectangle}\\ &= \text{length} \times \text{breadth}\\ &= (\pi \times \text{radius}) \times \text{radius}\\ &= \pi \times (\text{radius})^2\end{aligned}$$

Example 3

Find the area of a circle whose radius is 10 cm.

Solution

$$\begin{aligned}\text{Area of circle} &= \pi \times (\text{radius})^2\\ &= \pi \times 10^2\ \text{cm}^2\\ &\simeq 314\ \text{cm}^2\end{aligned}$$

Using $\pi \simeq 3.14$.

The course calculator has a π key. This key uses a much more accurate value for π than the approximate value 3.14 quoted above. You may choose to use your calculator, with its accurate π value, rather than the approximate value 3.14 in the exercise below.

Use of the π key is explained in Section 1.7 of Chapter 1 of the *Calculator Book*.

Solutions on page 92.

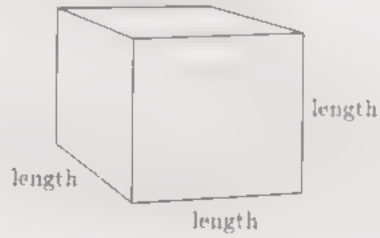
Try some yourself (7.1.3)

1 Find the area of a circle of (a) radius 7 cm and (b) radius 21 m.

7.1.4 Volumes

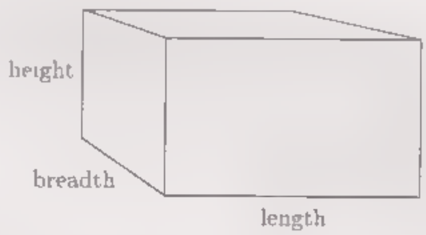
A **cube** is a solid object each of whose faces is a square.

The volume of a cube is the cube of the length of one of its edges.



Volume of cube = length × length × length = $(\text{length})^3$

The volume of a rectangular box is its length times its breadth times its height.

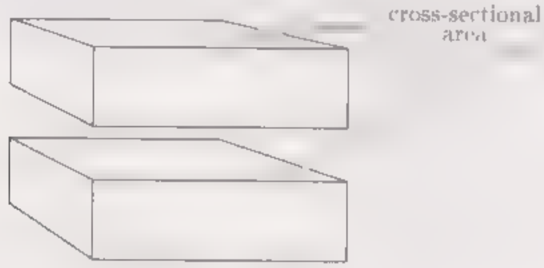


Volume of box = length × breadth × height

But the length × breadth is just the area of the bottom (or top) of the box, and so an alternative formula is:

Volume of box = area of base × height

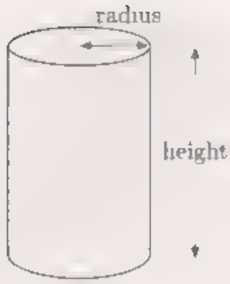
This formula is valid for other shapes which are similar to the box in one important respect. If you slice through the box horizontally at any point, the sliced area is the same as the area of the base. This is called the **cross-sectional area**.



For objects that have a constant cross-sectional area, we have:

Volume = cross-sectional area × height

This formula can be used to find the volume of a cylinder.

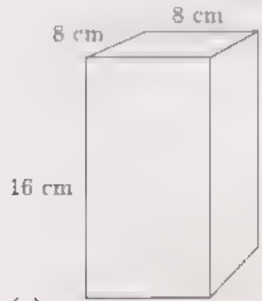


The cross-sectional area of a cylinder is just the area of the circle that forms its base. Therefore we have:

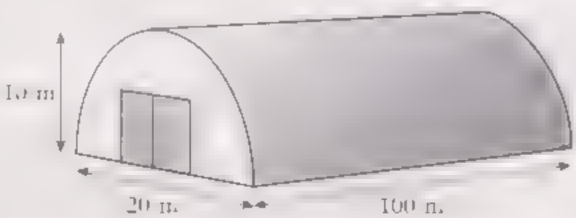
Volume of cylinder $= \pi \times (\text{radius})^2 \times \text{height}$

Example 4

Find the volumes of the following:



(a)



(b)

Solution

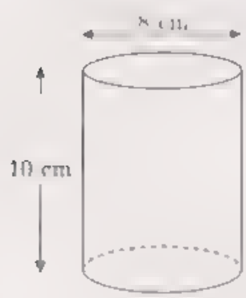
(a) Cross-sectional area $= 8 \times 8\,\text{cm}^2 = 64\,\text{cm}^2$
Volume$= 16 \times 64\,\text{cm}^3 = 1024\,\text{cm}^3$

(b) Cross-sectional area $=$ area of semicircle $= \frac{1}{2} \times \pi \times (\text{radius})^2$
$= \frac{1}{2} \times \pi \times 10^2\,\text{m}^2 \simeq 157\,\text{m}^2$
Volume $= 157 \times 100\,\text{m}^3 = 15\,700\,\text{m}^3$

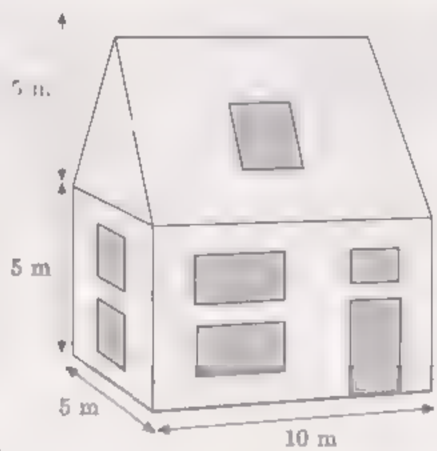
Try some yourself (7.1.4)

Solutions on page 92.

1 Find the volumes of the following



(a)



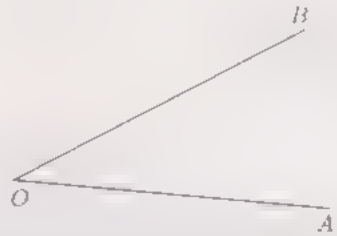
(b)

7.2 Angles

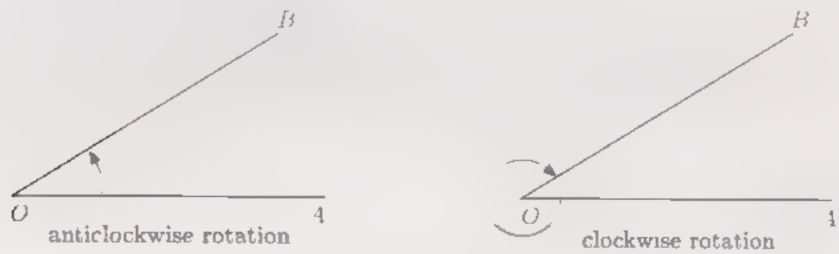
7.2.1 Angles, notation and measurement

The word 'angle' comes from the latin *angulus* meaning corner.

An **angle** is defined as the inclination of two straight lines to each other. Alternatively, you can think of an angle as the space at the corner or **vertex** of two straight lines.



More precisely, the angle between two straight lines is the amount of rotation required to take one of the lines to the other. But even this definition is ambiguous because rotation may be clockwise or anticlockwise. Think of O as a pivot and move OA until it lies exactly on top of OB. The diagram below shows that if OA is rotated anticlockwise a small rotation is all that is required to bring it to OB. However, if OA is rotated clockwise it needs to move through almost a complete circle before it reaches OB.



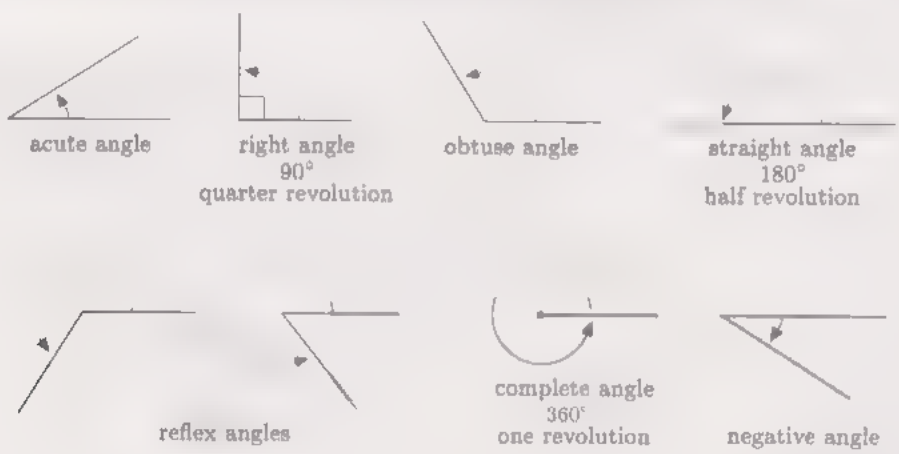
Angles measured clockwise are considered to be *negative* angles.

Angles can also be measured in *radians*.

To overcome this ambiguity angles are always measured using anticlockwise rotations. Thus the angle between OA and OB is the amount of *anticlockwise* rotation which takes OA to OB.

Angles are often measured in degrees, denoted by °, with a complete turn or revolution equal to 360°.

There are seven different types of angle, all of which we come across in everyday life. Each of these angles is described below.



Acute angle – this angle is less than a quarter revolution (90 °). An example of an acute is angle is the angle a door makes when it is ajar.

Right angle – a quarter revolution: this angle is exactly 90 °. This angle can be described as a quarter turn. The angles at the corners of most doors, tiles and windows are all right angles.

Obtuse angle – this angle is between 90 ° and 180 °. An example of an obtuse angle is the angle between the cutting blades of a pair of scissors when they are open as widely as possible.

Straight angle – half a revolution: this angle is exactly 180 °. A flat open book forms a straight angle.

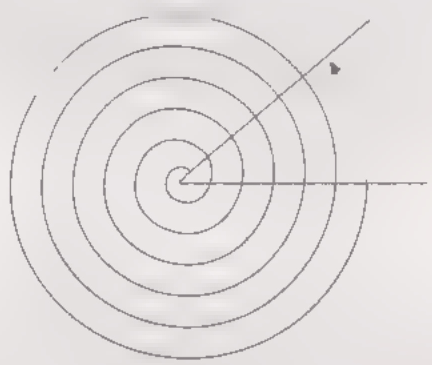
Reflex angle – this angle is between 180 ° and 360 °. When a box is opened and the lid goes right over to rest on the ground, the angle it sweeps out is reflex.

Complete angle – one revolution: this angle is exactly 360 °. This is the angle that the minute hand turns through every hour.

Negative angle – angle measured clockwise.

Angles can be greater than 360 ° for more than one complete revolution. For example, consider a helter-skelter and the angle about the centre for somebody going down it. Each complete revolution is 360 °. So six revolutions is $6 \times 360° = 2160°$.

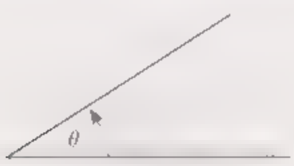
helter-skelter
six revolutions

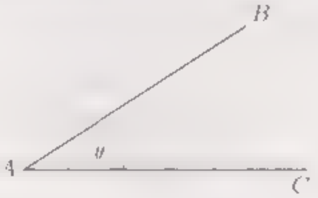


gives an angle of 2160°

There are several different notations used for angles. One notation is to use Greek letters (together with arrows to denote direction). Very often the arrow may be omitted, if there is no ambiguity as to the direction.

The Greek alphabet is listed in the Appendix

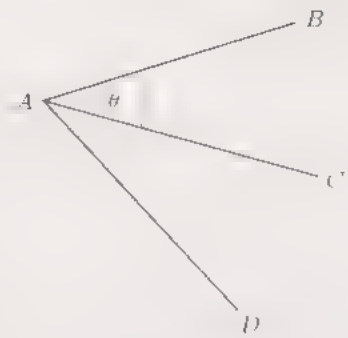




Alternatively an angle may be denoted by the vertex's label with a hat on it, so that for example, the angle θ above may be denoted A.

$\widehat{A}$ is pronounced 'angle A'

However, this notation can be ambiguous if there is more than one angle at the vertex A, as in the example below.



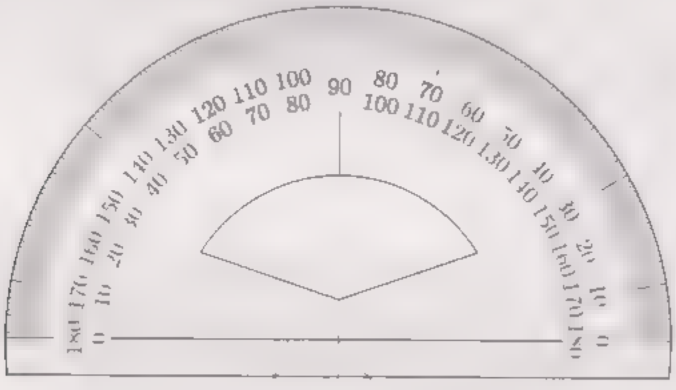
Both $C\hat{A}B$ and $\angle CAB$ are read as 'angle CAB'.

In such cases, we can denote θ by $C\hat{A}B$ or $\angle CAB$. Here, the middle letter indicates the vertex and the two outer letters identify the 'arms' of the angle.

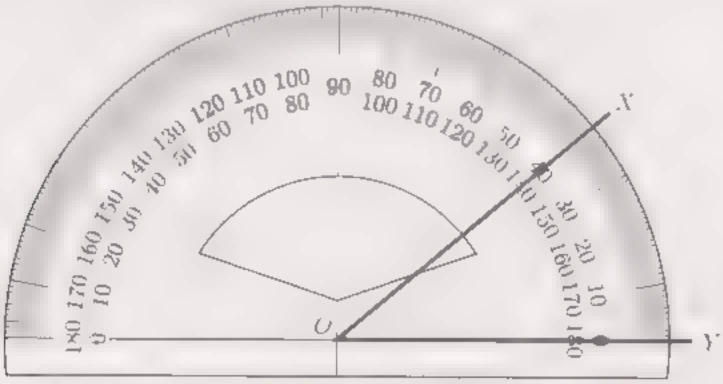
A right angle is often denoted by drawing a square between the arms of the angle.

If the angle between two straight lines is 90° then the lines are said to be **perpendicular** to each other.

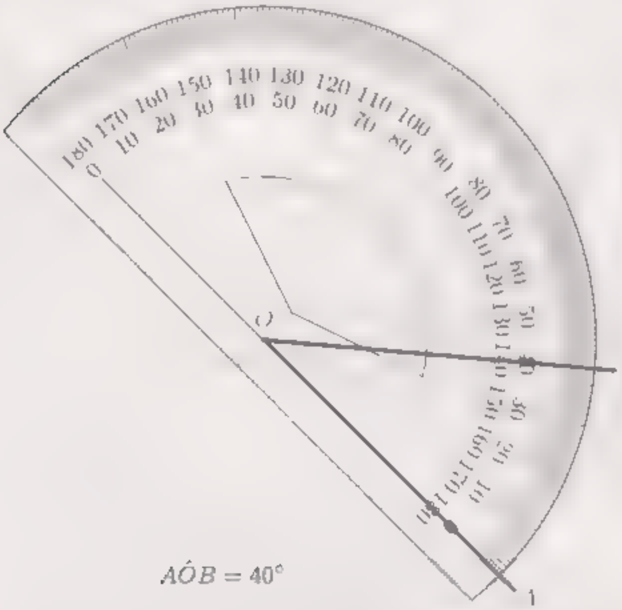
To measure an angle you need a protractor. This protractor below is graduated to measure angles from 0° to 180°. It is possible to buy circular protractors which measure angles from 0° to 360°.



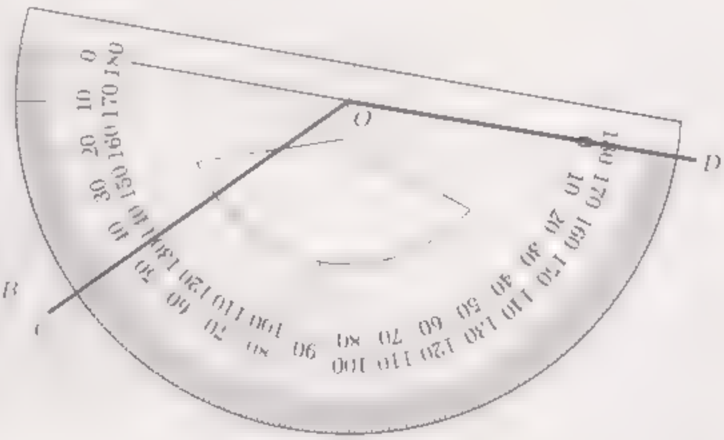
The following diagram indicates how the protractor should be positioned in order to measure an angle. Place the base-line of the protractor on one arm of the angle with the centre, O, on the vertex. The angle can then be read straight from the scale. Thus $Y\hat{O}X = 40°$.



In the above example, one of the arms of the angle is horizontal. However, you may find that you need to position the protractor in an awkward position in order to measure the angle. But be careful to read off from the correct scale.



$A\hat{O}B = 40°$



$C\hat{O}D = 135°$

You can also use a protractor to draw an angle, but once you have drawn the angle be on the safe side and measure it to check that it is correct.

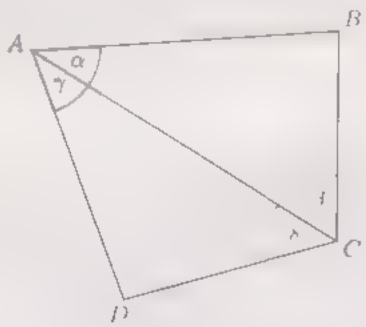
Try some yourself (7.2.1)

Solutions on page 92.

1 What angle has a roundabout gone through when it has turned through four revolutions (anticlockwise)?

2 Give an alternative notation for labelling each of the following angles, in the diagram below.

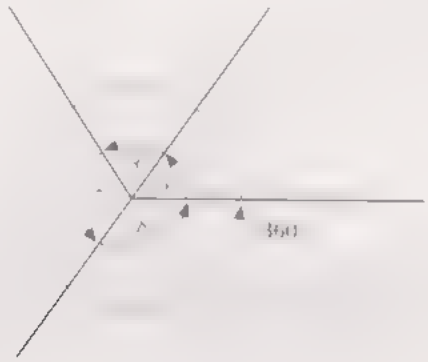
(a) α (b) β (c) $D\hat{A}C$ (d) $\angle ACD$



3 Use your protractor to measure each of the following angles, and state whether the angle is acute, right, obtuse, etc.

7.2.2 *Angles, points and lines*

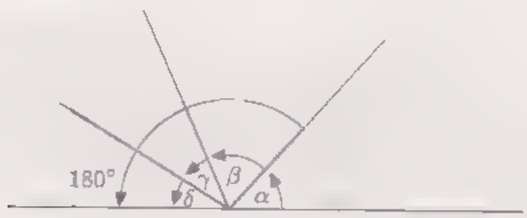
One complete revolution is $360°$. So if you turn through a number of angles and in total you have completed one revolution, the sum of the angles through which you have turned must be $360°$. So, in the diagram below, $\alpha + \beta + \gamma + \delta = 360°$.



Whenever you have a number of angles at a point like this they must add up to $360°$.

The sum of angles at a point is $360°$.

Similarly if you turn through several angles and in total do half a revolution then the sum of the angles through which you have turned must add up to half a revolution, i.e. $\frac{1}{2} \times 360° = 180°$. So, in the diagram below, $\alpha + \beta + \gamma + \delta = 180°$.



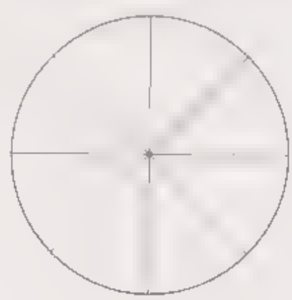
Whenever you have angles on a straight line their sum must be $180°$.

The sum of angles on a line is $180°$.

You can sometimes use these properties to determine unknown angles.

Example 5

Calculate the angle between neighbouring spokes of this wheel.

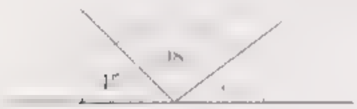


Solution

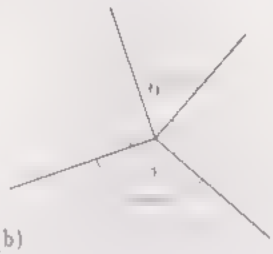
The eight spokes divide the circle up into eight equal parts. Thus the angle required is found by dividing $360°$ by 8 to give $45°$.

Example 6

Find α and β in the following diagrams.



(a)



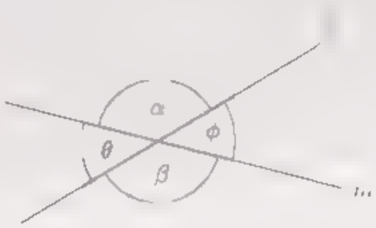
(b)

Solution

(a) $\alpha + 45° + 98° = 180°$
So $\alpha = 180° - 45° - 98° = 37°$.

(b) $90° + 60° + 90° + \beta = 360°$
So $\beta = 360° - 90° - 60° - 90° = 120°$.

In the diagram below, there are two intersecting straight lines, l and m, and four angles, α, θ, β and ϕ. Notice that the angles α and β are opposite each other; more precisely they are said to be **vertically opposite**. Similarly, θ and ϕ are vertically opposite angles.



In fact $\alpha = \beta$ and $\theta = \phi$. This can be proved using the fact that l and m are straight lines.

$\alpha + \theta = 180°$ since l is a straight line.
$\theta + \beta = 180°$ since m is a straight line.

Hence $\alpha + \theta = \theta + \beta$, and so $\alpha = \beta$. Subtract θ from both sides.

A similar argument proves that $\theta = \phi$. Try proving this for yourself.

Thus, for two intersecting straight lines:

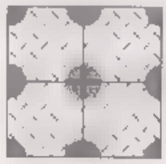
Vertically opposite angles are equal.

Therefore two intersecting straight lines result in two pairs of equal angles.

Solutions on page 92.

Try some yourself (7.2.2)

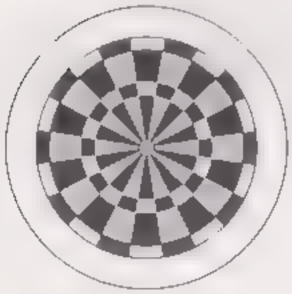
1 Calculate each angle at the centre of these objects.



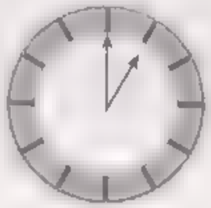
(a) Floor tiles

(b) Steering-wheel

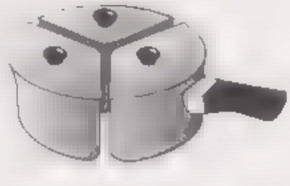
(c) Needlework box



(d) Dart-board



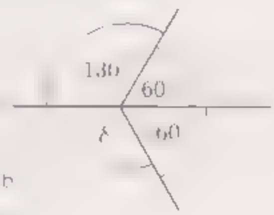
(e) Clock



(f) Set of pans

2 Find γ and δ in the following diagrams.

(a)

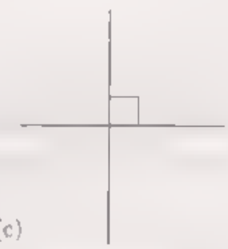


(b)

3 Find all the remaining angles in each of the following diagrams:

(a)

(b)

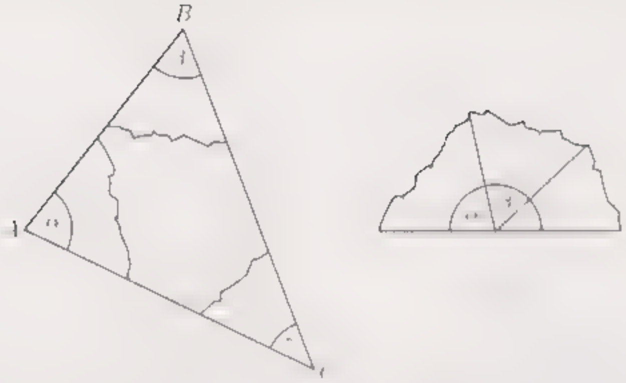


(c)

7.2.3 *The angles of a triangle*

The sum of the angles of any triangle is 180°. This property can be demonstrated in several ways. One way is to draw several triangles, measure the angles with a protractor and add them up. Another way is to draw a triangle on a piece of paper, mark each angle with a different symbol and cut off the angles and arrange them side by side touching one another as shown at the top of the next page.

A third way is to consider the angles that you turn through when you walk round a triangle. Suppose you start at A and walk along AB, then along



BC, then back along CA and finally turn to face along AB. You will then have turned through the following angles each time you changed direction at a vertex:

at B you turned through $180 - \beta$
at C you turned through $180 - \gamma$
at A you turned through $180 - \alpha$

So in total you have turned through

$$180° - \beta + 180° - \gamma + 180° - \alpha.$$

However, in total you have also turned through one revolution, and so

$$180° - \beta + 180° - \gamma + 180° - \alpha = 360°$$

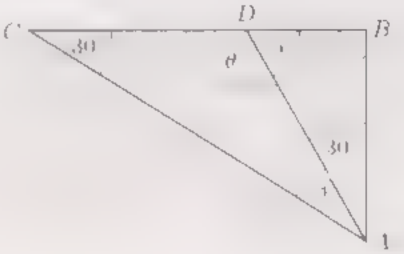
Therefore, subtracting 360° from both sides, we have

$$180° - \alpha - \beta - \gamma = 0, \quad \text{and hence} \quad \alpha + \beta + \gamma = 180°.$$

The fact that the sum of the angles of a triangle add up to 180° sometimes enables you to find unknown angles.

Example 7

Find α, β and θ in the diagram below.



Solution

Consider ΔABD. $D\hat{B}A = 90°$ and $B\hat{A}D = 30°$. Thus $\alpha + 30° + 90° = 180°$, and so $\alpha = 60°$.

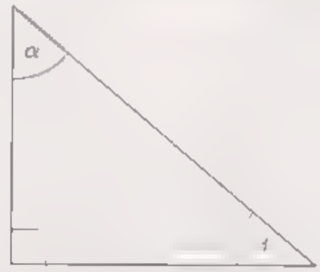
CDB is a straight line. So $\theta + \alpha = 180°$, and hence $\theta = 180° - \alpha = 180° - 60° = 120°$.

Now consider ΔADC. $A\hat{C}D = 30°$ and $C\hat{D}A = \theta = 120°$. Thus $\beta + 30° + 120° = 180°$, and so $\beta = 30°$.

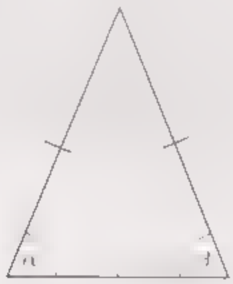
Check for yourself that the angles of ΔABC also add up to 180°.

You can also use the angle sum property of triangles together with other facts about particular types of triangle to deduce facts about the angles of such triangles.

In a *right-angled triangle*, since one angle is a right angle (90 °), the other two must add up to 90 °. So in the example below $\alpha + \beta = 90°$.



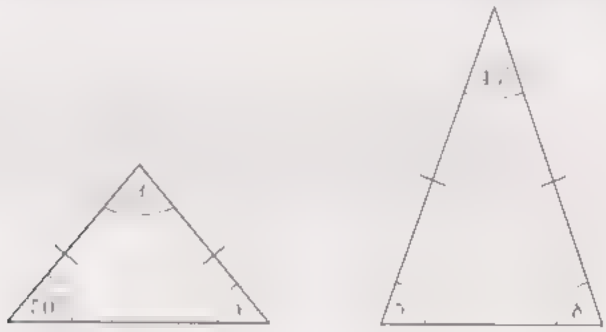
In an *isosceles triangle*, two sides are of equal length. So, drawing in the line of symmetry shows that the **base angles** α and β are both the same: $\alpha = \beta$.



Therefore, if one angle of an isosceles triangle is known, it is possible to calculate the other two.

Example 8

Find the unknown angles in the following isosceles triangles.



(a) (b)

Solution

(a) α and 50 ° are the base angles, so $\alpha = 50°$.

$$\beta = 180 - 50° - 50° = 80°.$$

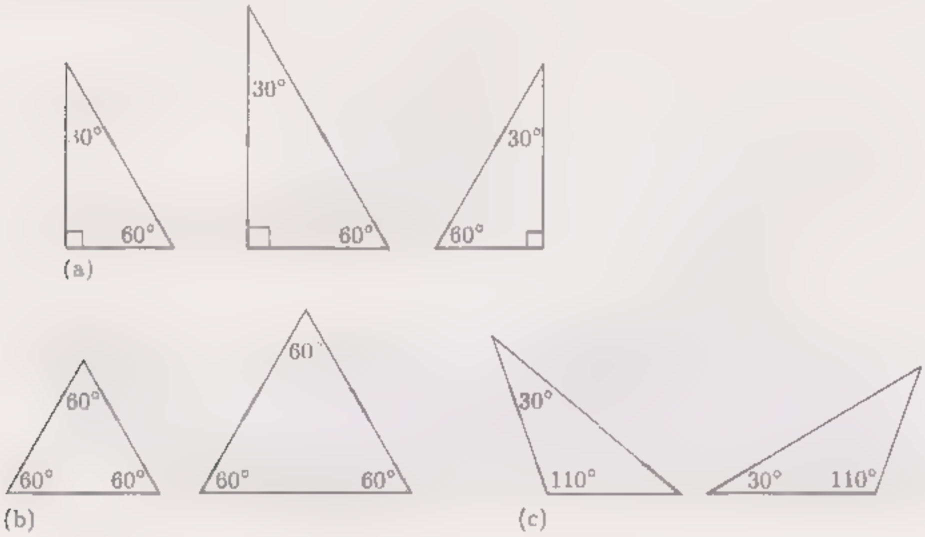
(b) γ and δ are the base angles, so $\gamma = \delta$. By the angle sum property of triangles,

$$40° + \gamma + \delta = 40° + 2\gamma = 180°.$$

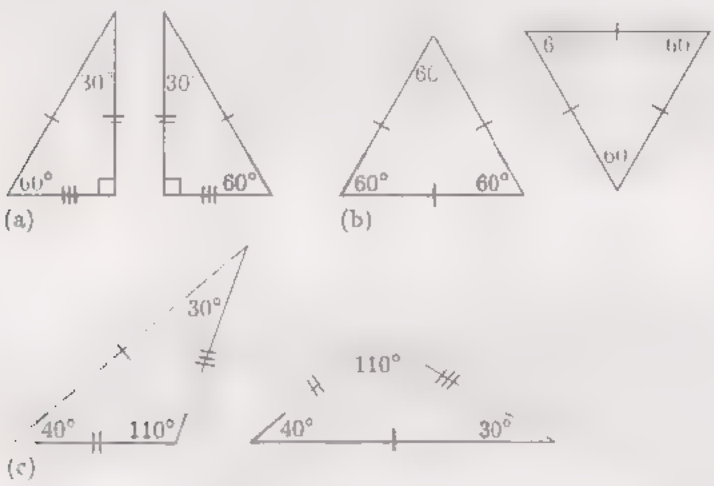
Therefore $2\gamma = 140°$, and hence $\gamma = \delta = 70°$.

In an *equilateral triangle*, all sides are of equal length. So, by symmetry, all angles are equal too. Hence each angle of an equilateral triangle is $180° \div 3 = 60°$.

Two triangles are said to be **similar** if they have the same shape. One triangle is a scaled version and/or a reflection and/or a rotation of the other. Hence the corresponding angles must be the same in similar triangles. In fact, if you know that two corresponding angles are equal, the third will be too, as the sum must be 180°. Three sets of similar triangles are shown below.



If two triangles are the same shape *and* the same size, they are said to be **congruent**. One triangle is a reflection and/or reflection and/or a rotation of other. Hence, as in the case of similar triangles, the corresponding angles must be the same in congruent triangles. Three pairs of congruent triangles are shown below.

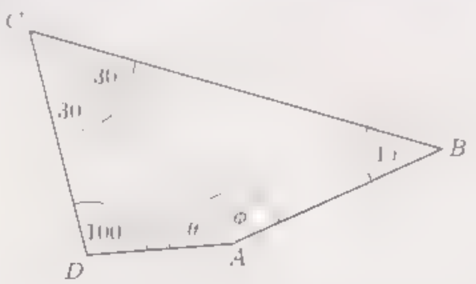


Sides of the same length are marked with the same number of short lines.

The various angle properties can also be used to find the sum of the angles of a quadrilateral.

Example 9

$ABCD$ is a quadrilateral. Find θ and ϕ, and thus the sum of all the angles of the quadrilateral.



Solution

$\phi = 180° - 30° - 40° = 110°$ $\theta = 180° - 100° - 30° = 50°$

Therefore the sum of all the angles of the quadrilateral is

$40° + (30° + 30°) + 100° + (50° + 110°) = 360°.$

In fact all quadrilaterals can be split into two triangles in the above way, and so their angles always add up to 360°.

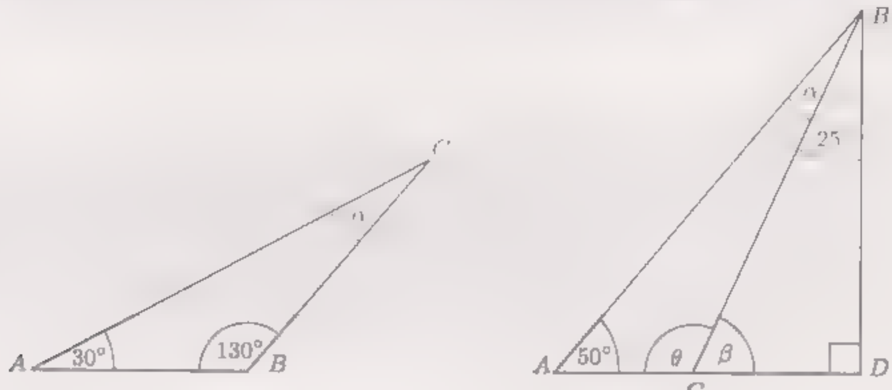
> The sum of the angles of a quadrilateral is 360°.

Similarly other polygons (i.e. shapes with straight sides) can be divided into triangles to find the sum of their angles.

Solutions on page 92.

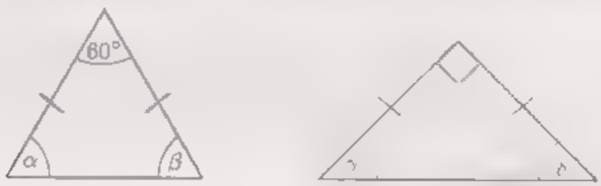
Try some yourself (7.2.3)

1 Find the unknown angles in each of the following diagrams.



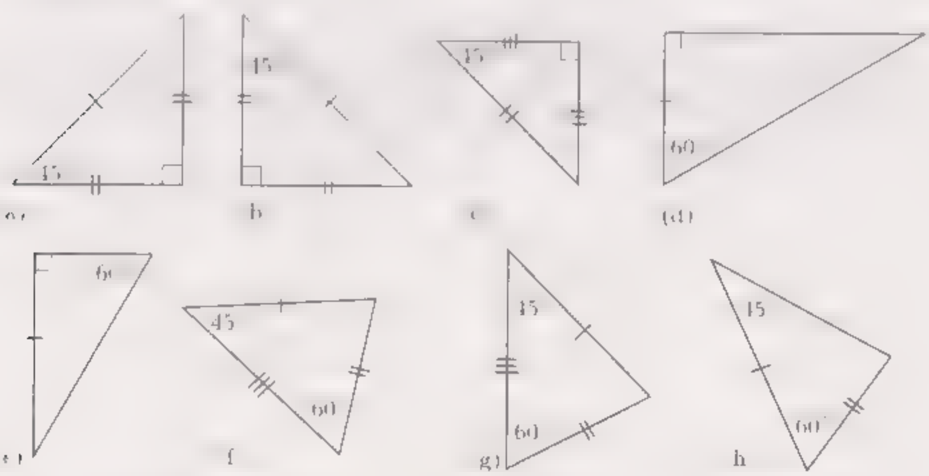
(a) (b)

2 Find the unknown angles in the following isosceles triangles.



(a) (b)

3 Which of the triangles below are similar to each other, and which are congruent to each other?



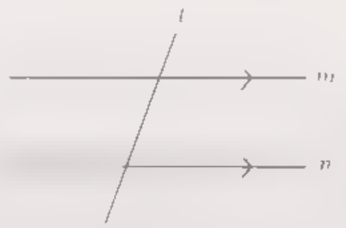
7.2.4 *Parallel lines*

Two straight lines which do not intersect, no matter how far they are extended, are said to be **parallel**.

Arrows are used to indicate parallel lines.

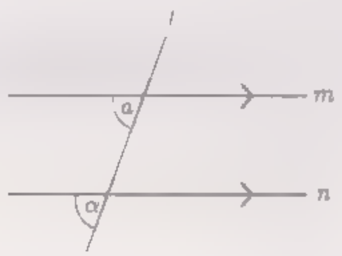
It's very difficult to determine whether or not two lines are parallel just by inspection. But there are two angle properties that provide reliable tests.

The first of these properties concerns what are called *corresponding angles*. Consider a line l, which intersects two parallel lines m and n.



If you trace one of the intersections and place it over the other, you will find that the lines coincide exactly. The four angles at each intersection also coincide exactly. Thus $\alpha = a$, $\beta = b$, $\gamma = c$ and $\delta = d$.

The pairs of angles which correspond to each other in such intersections are called **corresponding angles**. For example, α and a below are corresponding angles. Because m and n are parallel, α and a are equal.

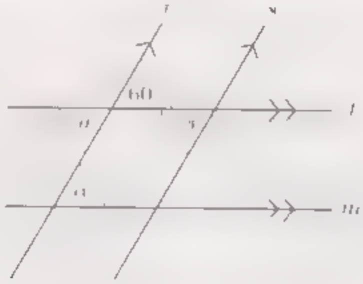


So, when a line intersects two parallel lines, corresponding angles are equal. Also, if a line intersects two lines and corresponding angles are *not* equal, then the lines are *not* parallel; but if corresponding angles *are* equal, the relevant lines *are* parallel. This property can be used to determine whether or not two lines are parallel. It can also be used to find unknown angles.

Example 10

Find α and β in the diagram below.

r and s are parallel lines, indicated by the single arrows; l and m are also parallel, indicated by the double arrows.

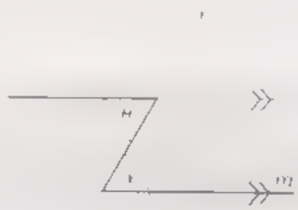


Solution

l is parallel to m, so $\alpha = 60°$ (corresponding angles).

r is parallel to s, so $\theta = \beta$ (corresponding angles). But $\theta = 60°$ (vertically opposite angles). Thus $\beta = 60°$.

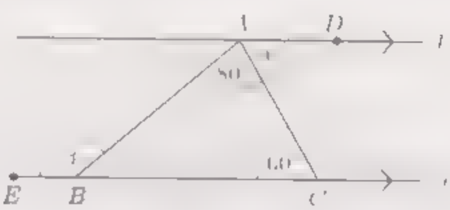
The second property is illustrated by the example above. Notice that $\alpha = 60°$ and $\theta = 60°$. These two angles are those of a Z shape, indicated by the solid line in the diagram below.



Such angles are called **alternate angles**. So, when a line intersects two parallel lines, alternate angles are equal. Also, if a line intersects two lines and alternate angles are *not* equal, then the lines are *not* parallel; but if alternate angles *are* equal, the relevant lines *are* parallel. Again, this property can be used to determine whether or not two lines are parallel, as well as to find unknown angles.

Example 11

Find α and β in the diagram below.



Solution

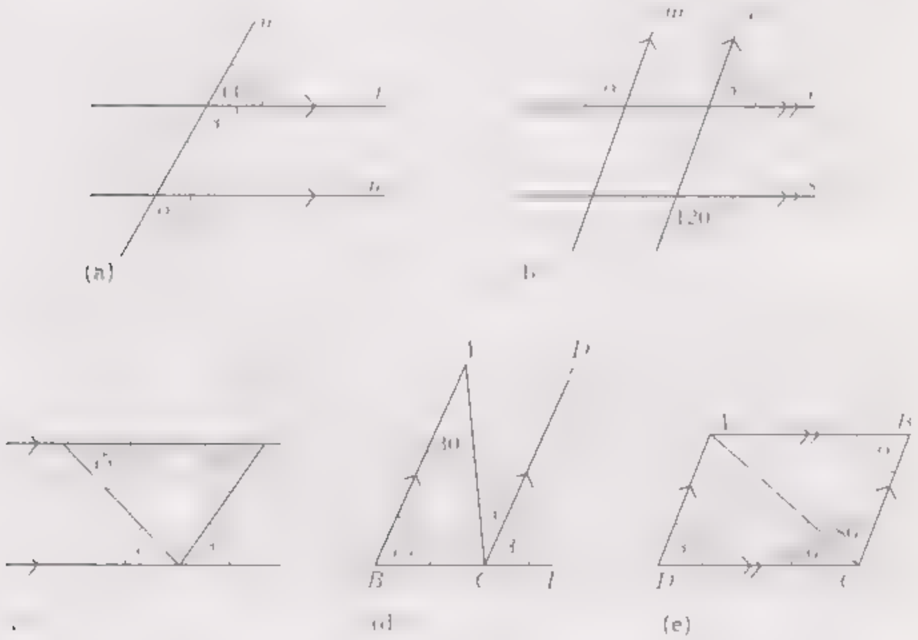
l is parallel to m, so $C\hat{A}D = A\hat{C}B$ (alternate angles).
Thus $\alpha = C\hat{A}D = 60°$.

Now, $B\hat{A}D = 80° + \alpha = 80° + 60° = 140°$,
and $A\hat{B}E = B\hat{A}D$ (alternate angles).
Thus $\beta = A\hat{B}E = 140°$.

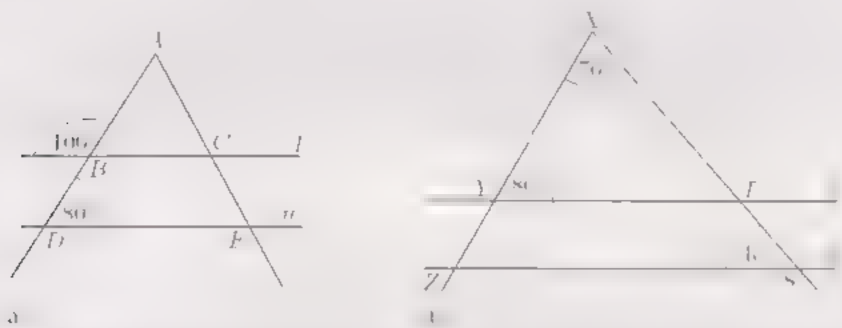
Try some yourself (7.2.4)

Solutions on page 93.

1 Find α and β in each of the diagrams below.



2 Decide whether the lines l and m are parallel in each of the following diagrams.



Outcomes

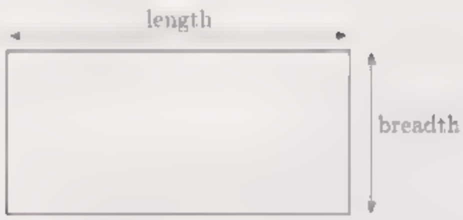
Now that you have studied Module 7 you should be able to:

- ◇ determine the lines of symmetry, the centre of rotation and the order of rotational symmetry of simple shapes
- ◇ understand what is meant by a quadrilateral, rectangle, square, parallelogram, trapezium, triangle, right-angled triangle, isosceles triangle, equilateral triangle and scalene triangle
- ◇ find the areas of triangles and quadrilaterals of all types
- ◇ understand what is meant by a circle, circumference, radius, diameter, semicircle and sector
- ◇ find the area and circumference of a circle
- ◇ find the areas of shapes based on triangles, quadrilaterals and circles
- ◇ find the volumes of boxes, cylinders and other shapes with constant cross-sectional areas
- ◇ understand what is meant by an angle and a vertex, and determine whether an angle is acute, right, obtuse, straight, reflex, complete or negative
- ◇ measure angles in degrees and use appropriate notation for angles
- ◇ state the sum of the angles at a point, on a line, of a triangle and of a quadrilateral, and use these facts to determine unknown angles
- ◇ recognize similar and congruent triangles
- ◇ understand what are meant by parallel lines
- ◇ recognize vertically opposite, corresponding and alternate angles and use facts about these to determine unknown angles and whether lines are parallel

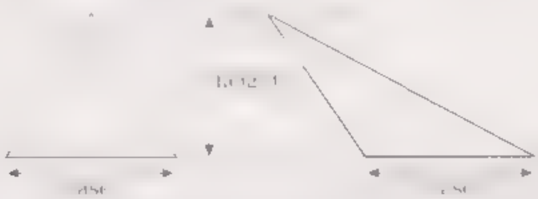
Appendix Formulas and Greek letters

Formulas

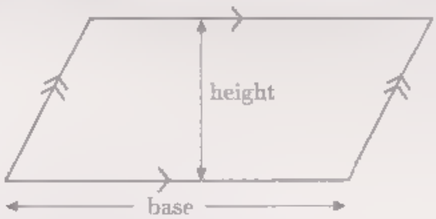
Area of rectangle = length × breadth.



Area of triangle = $\frac{1}{2}$ × base × perpendicular height.



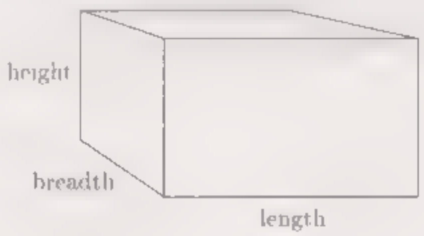
Area of parallelogram = base × perpendicular height.



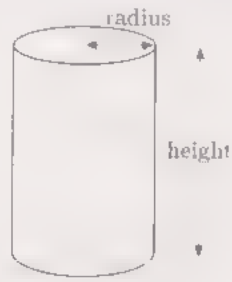
Area of circle = $\pi \times (\text{radius})^2$. $\pi \simeq 3.14$.

Circumference of circle = 2π × radius.

Volume of box = length × breadth × height
= area of base × height.



Volume of cylinder $= \pi \times (\text{radius})^2 \times$ height.



The Greek alphabet

A	α	Alpha
B	β	Beta
Γ	γ	Gamma
Δ	δ	Delta
E	ϵ	Epsilon
Z	ζ	Zeta
H	η	Eta
Θ	θ, ϑ	Theta
I	ι	Iota
K	κ	Kappa
Λ	λ	Lambda
M	μ	Mu
N	ν	Nu
Ξ	ξ	Xi
O	o	Omicron
Π	π	Pi
P	ρ	Rho
Σ	σ, ς	Sigma
T	τ	Tau
Υ	υ	Upsilon
Φ	ϕ, φ	Phi
X	χ	Chi
Ψ	ψ	Psi
Ω	ω	Omega

Solutions

Section 5.1

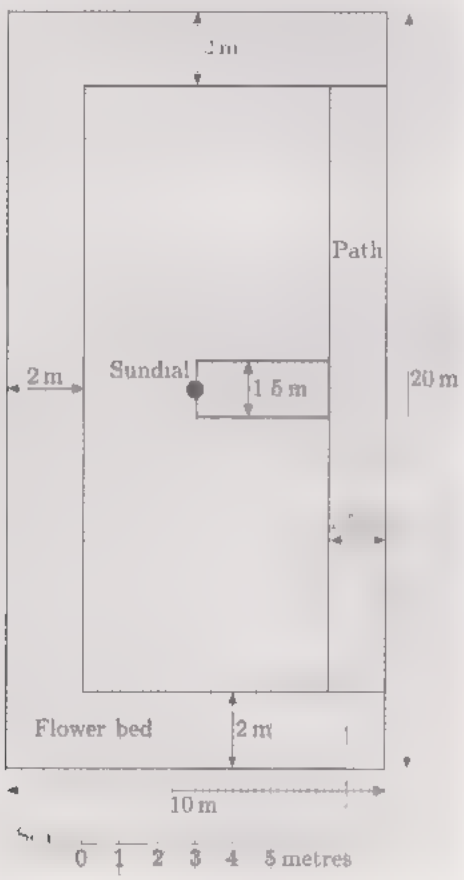
1 The window in the diagram is 1.1 squares wide, so in reality it is 1.1 m wide.

The wash basin in the diagram is $\frac{3}{5}$ of a square deep by $\frac{9}{10}$ of a square wide. So in reality it is 0.6 m by 0.9 m.

2 5 cm represents 1 m.
10 cm is 2×5 cm so represents 2 m.
20 cm is 4×5 cm so represents 4 m.
1 cm is $\frac{1}{5} \times 5$ cm so represents $\frac{1}{5}$ m or 0.2 m.

3 1 km is represented by 2 cm.
10 km is represented by $10 \times 2 = 20$ cm.
5 km is represented by $5 \times 2 = 10$ cm.
0.5 km is represented by $0.5 \times 2 = 1$ cm.

4 Your scale plan should look something like this:



Section 5.2.1

1

(a) 15 minutes.

(b) The temperature drops from 90 °C to 36 °C in the first 20 minutes, a total of 54 degrees. In the second 20 minutes, the temperature drops from 36 °C to 25 °C, a total of 11 degrees.

(c) The tea cools off quickly at the beginning, when it is hot, then cools more gradually, when it is cooler. (This is certainly my experience when drinking tea!)

2

(a) The Potters will fly on aircraft B737 leaving Birmingham at about 08.35 arriving back at about 14.55 one week later. The holiday number is T2255. The cost for a 6 October departure is £1200 per adult. Since their daughter is 12 she is not entitled to any reduction. The total cost of the holiday is therefore £3600. The flight time from Birmingham to Alicante is 2 hours 20 minutes and it takes about 1 hour 5 minutes to transfer from the airport to the hotel.

(b) The most expensive holiday costs £2360, for two weeks starting between 6 and 31 August, leaving from Glasgow.

The cheapest holiday is for one week leaving from Gatwick on 26 April at a cost of £980

(To glean information like this from a table it is necessary to go through the data systematically. Even then it is easy to make mistakes. Unfortunately there is no quick method; you just have to be careful.)

3

(a) The 100% base for 1977 was 11 979.

(b) 18% of the households surveyed in 1977 contained four people. The actual number was 18% of 11 979, which is 2156.22, or 2156 rounded to the nearest whole number. Therefore 2156 of the households surveyed in 1977 comprised four people.

(c) The percentage total for the 1977 column is

$$21 + 33 + 17 + 18 + 7 + 4 = 100\%.$$

4 34.9% of families consisted of a married couple with two dependent children. The total number of families surveyed was 4855. 34.9% of 4855 = 1694.395 = 1694 (rounded to the nearest whole number). So the actual number of families consisting of a married couple with two dependent children was 1694.

5

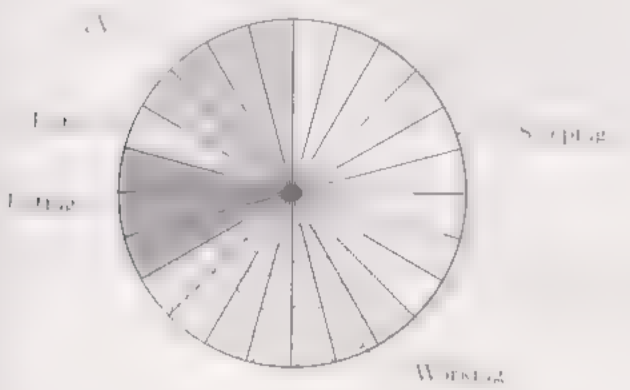
(a) 19%. (Find '25–34' in the 'Age' column: then look across this row to the 'under 20' column.)

(b) The '60 or over' row gives 60 as the percentage of men aged 60 or over who do not smoke and 2150 as the total number of men aged 60 or over in the sample. So 60% of 2150 = 1290 of the men aged 60 or over are non-smokers.

(c) Men of 16–24 and of 60 or over who smoke over 20 cigarettes per day constitute less than 20% of their age groups. In the 25–34, 35–49 and 50–59 age groups, well over 20% smoke over 20 cigarettes per day. So the heaviest smokers are in the age range 25–59, with those aged 25–34 being marginally the heaviest smokers.

Section 5.2.2

1 One way of shading and labelling the chart is:



2 The largest proportion of people (about 30%) live in semi-detached houses. Over 25% live in terraced houses and about 20% live in detached houses. About 15% live in flats or maisonettes and about 10% live in some other type of accommodation.

3

(a) Detached houses.

(b) Terraced houses.

(c) From the first pie chart, labelled 'all households', we see that most people live in semi-detached houses, although almost as many live in terraced houses.

(d) About 30% of employers/managers live in detached houses compared with only about 10% of skilled manual workers. Marginally more skilled manual workers than employers/managers live in semi-detached houses, though the proportions in this category (about 30%) are very similar. More skilled manual workers (about 30%) than employers/managers (about 15%) live in terraced houses. About the same proportion of each (about 25%) live in flats or maisonettes and 'other' accommodation.

Section 5.2.3

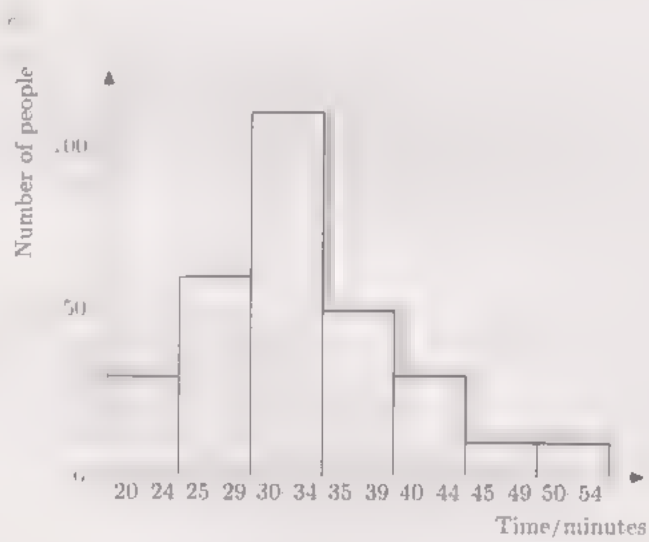
1

(a) About 12 000.

(b) About 21 000.

(c) Generally the figures are increasing, although there was a slight fall in 1974.

2



(b) Each interval is 5 minutes wide.

3

(a) About 10%.

(b) About 3%.

Section 5.3.1

1 A has coordinates $(2, 3)$.
B has coordinates $(^-1, 2)$.
C has coordinates $(^-2, ^-1)$.
D has coordinates $(1, ^-2)$.
E has coordinates $(^-2, 0)$.

2

3

Note that where a point does not plot exactly onto one of the grid lines, you have to estimate its correct position.

4

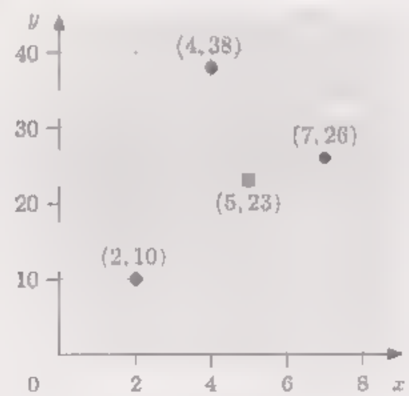
(a) $(9.5, 10)$. At 9.30 am the temperature was 10 °C.

(b) $(900, 1.4)$. The cost of 900 grams (0.9 kg) of washing powder is £1.40.

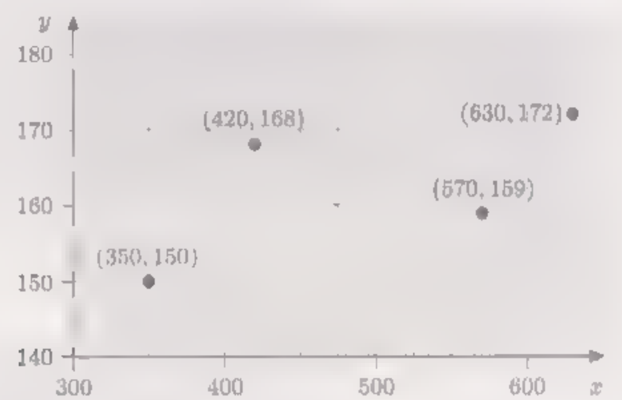
(c) $(24, ^-60)$. On the 24th of the month there was $^-$£60 in the bank (i.e. on the 24th of the month the account was overdrawn by £60).

5 The scales you chose will have depended very much on the size of your graph paper: you should have chosen ones that are easy to use for plotting the given points. Your solutions should be something like the following.

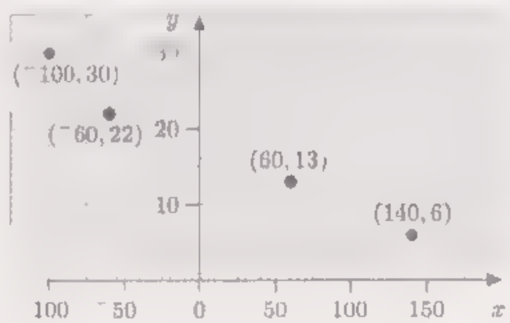
(a) x-axis: 1 large square: 2 units
y-axis: 1 large square: 10 units



(b) x-axis: 1 large square: 50 units
y-axis: 1 large square: 10 units

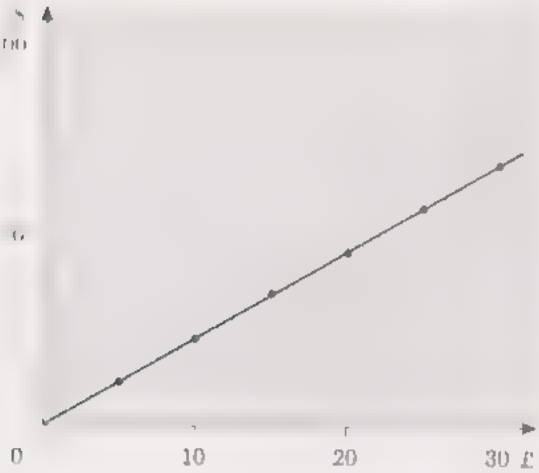


(c) x-axis: 1 large square: 50 units
y-axis: 1 large square: 10 units



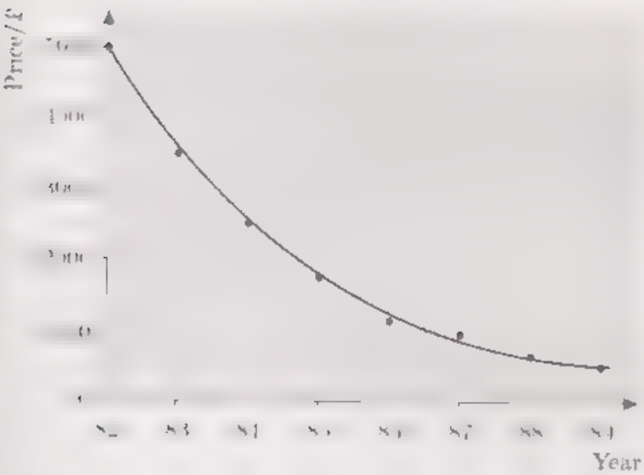
Section 5.3.2

1



2 The cost of coffee increased over the period from £5 per kilogram to over £10 per kilogram. It remained steady between February and April rose significantly during the month of April, then increased slowly but steadily until July.

3



4

(a) About 80%.

(b) About 90% earned more than £7500 a year, and about 70% earned more than £10 000. About 10% earned more than £20 000, about 5% more than £25 000 and virtually no one more than £30 000. This means that 30% earned less than £10 000, 60% between £10 000 and £20 000, and 10% more than £20 000.

(c) About 10% of people with no qualifications earned over £20 000 per year. About 15% of people with CSE grades 2–5, etc. earned over £20 000 per year. The remaining percentages are about 25% of people with O-level or equivalent, 30% of people with A-level or equivalent, 55% of people with higher education below degree standard and about 80% of people with a degree or equivalent.

(d Most people with a degree earned over £20 000 (about 80%).

Most people with no qualifications earned under £20 000 (about 90%).

About 45% of people with a degree earned over £30 000, whereas virtually no one with no qualifications fell into this category.

The two graphs differ significantly over about £7500. The graph showing no qualifications drops quickly whereas the graph showing degree or equivalent falls off very gradually.

Section 6.1.1

1 These are not the only possibilities. You may have used other words, but check you have written in whole sentences with full stops at the end.

(a) $2.3 + 3.7 = 6$

and $14.8 - 5.6 = 9.2$.

Hence $\dfrac{2.3 + 3.7}{14.8 - 5.6} = \dfrac{6}{9.2}$

$= 0.65$ (rounded to two decimal places)

(b) $(3.2)^2 = 10.24$

and $(8.5)^2 = 72.25$.

So $(3.2)^2 + (8.5)^2 = 10.24 + 72.25 = 82.49$.

2 $2.3 + 3.7 = 6$ and $14.8 - 5.6 = 9.2$.

Hence $\dfrac{2.3 + 3.7}{14.8 - 5.6} = \dfrac{6}{9.2} = 0.652\,173\,913.$ (1)

Now $(3.2)^2 = 10.24$ and $(8.5)^2 = 72.25$.

So $(3.2)^2 + (8.5)^2 = 10.24 + 72.25 = 82.49$. (2)

So, using (1) and (2),

$$\frac{2.3 + 3.7}{14.8 - 5.6} + (3.2)^2 + (8.5)^2 = 0.652\,173\,193 + 82.49$$

$= 83.14$ (rounded to two decimal places).

Section 6.1.2

1 There are many different correct solutions to this exercise. However, make sure that your meaning is equivalent to the one given below.

Word	Mathematical meaning	Example of use
decimal	a number expressed in terms of tenths, hundredths, etc.	a quarter expressed as a decimal is 0.25
fraction	one whole number over another	0.75 expressed as a fraction is $\frac{3}{4}$
positive	greater than zero	2 is a positive number
negative	less than zero	$^-2$ is a negative number
scale	ratio between the size of something and a representation of it (e.g. a drawing or it is often graph); represented on a line or axis, which is also called the scale	the drawing had the scale indicated below it
triangle	a geometric figure with three straight sides	any three points define a triangle, provided they do not lie in a straight line
square	a geometric figure with four straight sides of equal length and right angles at the corners *or* the area of such a figure	a square of side 5 has an area of 25, so 25 is the square of 5
rectangle	a geometric figure with four straight sides and right angles at the corners	a square is a rectangle with sides of equal length

Section 6.1.3

1

(a) Add 5 and 8, and divide the result by the difference between 4 and 2 (giving $6\frac{1}{2}$).

(b) Add 5 to the quotient 8/4 and subtract 2 (giving 5).

(c) Multiply the sum of 4 and 5 by the difference between 5 and 2 (giving 27).

(d) 9 times the square root of 4 (giving 18).

(e) Three-quarters of 5 times 6 over 2 (giving $11\frac{1}{4}$).

(f) The cube of the square root of 25 (giving 125).

2

(a) The mass is greater than or equal to 10 kilograms.

(b) The time is less than 2.4 million hours.

(c) Two-thirds (or 2 divided by 3) is not equal to 0.67.

Section 6.2.1

1 The formula is

$$\text{cost of tomatoes} = (\text{price per kilogram}) \times (\text{number of kilograms}).$$

The (price per kilogram) is 75 pence and the (number of kilograms) is 1.45. So the formula gives

$$\begin{aligned}\text{cost of tomatoes} &= (\text{price per kilogram}) \times (\text{number of kilograms})\\ &= 75 \text{ pence} \times 1.45\\ &= 108.75 \text{ pence}.\end{aligned}$$

Rounded to the nearest penny this is 109 pence, or £1.09.

2 After 1.5 hours the formula gives

$$\begin{aligned}\text{distance travelled} &= (\text{average speed}) \times (\text{time taken})\\ &= 60 \times 1.5\,\text{km} = 90\,\text{km}.\end{aligned}$$

2 hours 40 minutes is $2\frac{2}{3}$ hours, or $\frac{8}{3}$ hours, so the formula gives

$$\begin{aligned}\text{distance travelled} &= (\text{average speed}) \times (\text{time taken})\\ &= 60 \times \tfrac{8}{3}\,\text{km} = 160\,\text{km}.\end{aligned}$$

After three and a half hours (3.5 hours) the formula gives

$$\begin{aligned}\text{distance travelled} &= (\text{average speed}) \times (\text{time taken})\\ &= 60 \times 3.5\,\text{km} = 210\,\text{km}.\end{aligned}$$

3 For the first room, with a wall area of 56 square metres,

$$\text{number of tins} = \frac{\text{area of wall}}{\text{area covered by one tin}} = \frac{56}{15} = 3\tfrac{11}{15} \text{ tins.}$$

For the second room, with a wall area of 38 square metres,

$$\text{number of tins} = \frac{\text{area of wall}}{\text{area covered by one tin}} = \frac{38}{15} = 2\tfrac{8}{15} \text{ tins.}$$

For the third room, with a wall area of 40 square metres,

$$\text{number of tins} = \frac{\text{area of wall}}{\text{area covered by one tin}} = \frac{40}{15} = 2\tfrac{2}{3} \text{ tins.}$$

The total number of tins required depends on whether you are planning to paint all the rooms in the same colour or not. If you are, the total number of tins required is

$$\tfrac{56}{15} + \tfrac{38}{15} + \tfrac{40}{15} = \tfrac{134}{15} = 8\tfrac{14}{15},$$

so you'd need to buy 9 tins. If all the rooms are to be different colours, you'll need to buy 4 tins for the first room, 3 for the second and 3 for the third—a total of 10 tins.

Section 6.2.2

1 Since 1 mile = 1.6093 km,
12 miles = 12 × 1.6093 km = 19.3116 km.

2 2.54 cm = 1 inch, so

1 cm = 1/2.54 inches
= 0.3937 inches
(rounded to four decimal places).

453.59 g = 1 lb, so

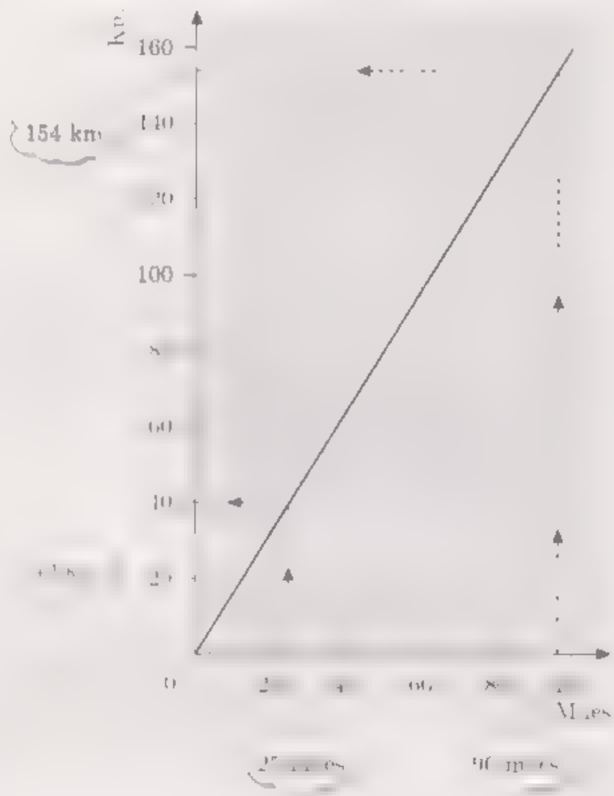
1 g = 1/453.59 lb
= 0.0022 lb (rounded to four decimal places).

Since 1 kg is 1000 g, this last formula leads to the formula

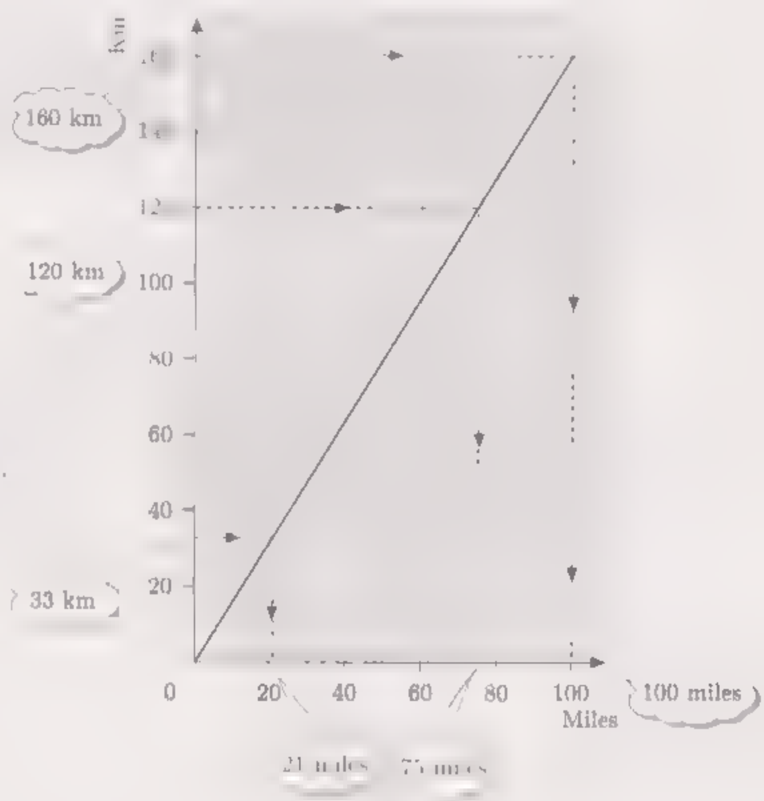
1 kg = 2.2 lb.

Section 6.2.3

1 From the diagram below, 25 miles is 40 km and 96 miles is 154 km.

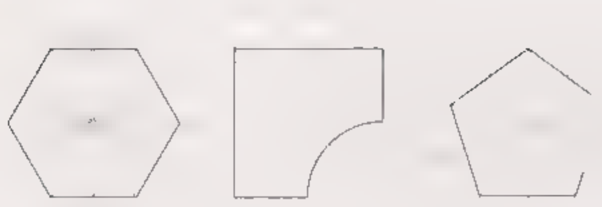


2 From the diagram below, 160 km is 100 miles, 120 km is 75 miles and 33 km is 21 miles.



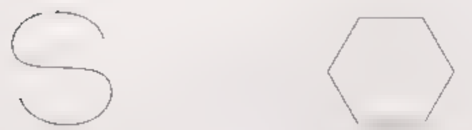
Section 7.1.1

1



(a) 6 lines of symmetry (b) 1 line of symmetry (c) 5 lines of symmetry

2



(a) order 2 (b) order 6

(c) order 5 (d) order 1

Notice that (d) has no rotational symmetry and no centre of notation. We say that it has rotational symmetry of order 1, as it requires a complete revolution (about any point) for it to look the same and lie in the same position.

3

(a) 8 lines of symmetry order 8 (b) 2 lines of symmetry order 2 (c) 6 lines of symmetry order 6

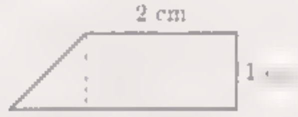
Section 7.1.2

1

(a) Area of triangle $= \frac{1}{2} \times$ base $\times$ height
$= \frac{1}{2} \times 8 \times 6\,\text{cm}^2 = 24\,\text{cm}^2$.

(b) Area of parallelogram = base × height
$= 3.4 \times 2.2\,\text{cm}^2 = 7.48\,\text{cm}^2$.

(c) The trapezium can be split into a triangle and a rectangle:



Area of trapezium = area of triangle + area of rectangle
$= (\frac{1}{2} \times 1 \times 1) + (2 \times 1)\,\text{cm}^2$
$= 2.5\,\text{cm}^2$.

2

Area front = area back $= 40 \times 25\,\text{cm}^2 = 1000\,\text{cm}^2$

Area near side = area far side $= 25 \times 25\,\text{cm}^2 = 625\,\text{cm}^2$

Area top = area bottom $= 40 \times 25\,\text{cm}^2 = 1000\,\text{cm}^2$

Total area $= 2 \times 1000 + 2 \times 625 + 2 \times 1000\,\text{cm}^2 = 5250\,\text{cm}^2$.

3

Area floor $= 5 \times 4\,\text{m}^2 = 20\,\text{m}^2$

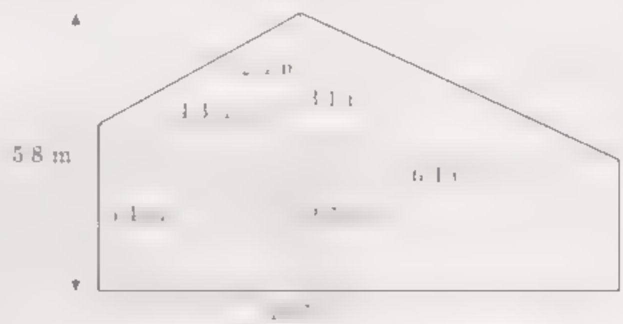
Area carpet $= 6\,\text{m}^2$

Area to be varnished $= 20 - 6\,\text{m}^2 = 14\,\text{m}^2$

Number of tins $= \frac{14}{2.5} = 5.6$

So 6 tins will be needed.

4 The end wall of the bungalow, minus the windows, can be divided into simple shapes as shown.



Area left triangle $= \frac{1}{2} \times 4.3 \times 2.4\,\text{m}^2 = 5.16\,\text{m}^2$

Area left rectangle $= 4.3 \times 3.4\,\text{m}^2 = 14.62\,\text{m}^2$

Area right triangle $= \frac{1}{2} \times 6.4 \times 3.1\,\text{m}^2 = 9.92\,\text{m}^2$

Area right rectangle $= 6.4 \times 2.7\,\text{m}^2 = 17.28\,\text{m}^2$

Total area of end wall = sum of areas above
$= 46.98\,\text{m}^2$

The dimensions of the windows, in metres, are 2.2 m by 1.45 m and 1.25 m by 0.88 m.

Area windows $= 2.2 \times 1.45 + 1.25 \times 0.88\,\text{m}^2$
$= 3.19 + 1.1\,\text{m}^2 = 4.29\,\text{m}^2$

Area to be painted = total area − area windows
$= 46.98 - 4.29\,\text{m}^2 = 42.69\,\text{m}^2$

5 Area one sloping roof $= 2.12 \times 5\,\text{m}^2 = 10.6\,\text{m}^2$
Area one side $= 2 \times 5\,\text{m}^2 = 10\,\text{m}^2$

$$
\begin{aligned}
\text{Area front/back} &= \text{area rectangle} + \text{area triangle}\\
&= (3 \times 2) + (\tfrac{1}{2} \times 3 \times 1.5)\ \text{m}^2\\
&= 8.25\,\text{m}^2
\end{aligned}
$$

Area door $= 0.8 \times 1.75\,\text{m}^2 = 1.4\,\text{m}^2$

Total area of canvas

$$
\begin{aligned}
&= 2 \times \text{area sloping roof} + 2 \times \text{area side}\\
&\quad + 2 \times \text{area front/back} - \text{area door}\\
&= 2 \times 10.6 + 2 \times 10 + 2 \times 8.25 - 1.4\,\text{m}^2\\
&= 56.3\,\text{m}^2
\end{aligned}
$$

So $56.3\,\text{m}^2$ of canvas are needed.
(In practice, the amount needed will depend upon the width of the canvas and on how many joins are needed. It is likely that at least $60\,\text{m}^2$ will be needed.)

Section 7.1.3

1

(a)
$$
\begin{aligned}
\text{Area of circle} &= \pi \times (\text{radius})^2\\
&= \pi \times 7^2\,\text{cm}^2\\
&= 154\,\text{cm}^2
\end{aligned}
$$
(rounded to the nearest square centimetre)

(b)
$$
\begin{aligned}
\text{Area of circle} &= \pi \times (\text{radius})^2\\
&= \pi \times 21^2\,\text{m}^2\\
&= 1385\,\text{m}^2
\end{aligned}
$$
(rounded to the nearest square metre)

Section 7.1.4

1

(a)
$$
\begin{aligned}
\text{Cross-sectional area} &= \pi \times (\text{radius})^2\\
&\ \ \pi \times 4\ \text{cm}^2\\
&= 50.265\,\text{cm}^2
\end{aligned}
$$
(rounded to three decimal places).

Volume $= 50.265 \times 10\text{cm}^3 = 502.65\,\text{cm}^3$
So the volume is $503\,\text{cm}^3$ to the nearest cubic centimetre.
(If you used the approximate value of 3.14 for π you will have got a cross-sectional area of $50.24\,\text{cm}^2$ and a volume of $502.4\,\text{cm}^3$.)

(b)
$$
\begin{aligned}
\text{Cross-sectional area} &= \text{area square}\\
&\quad + \text{area triangle}\\
&= (5 \times 5) + (\tfrac{1}{2} \times 5 \times 5)\ \text{m}^2\\
&= 37.5\,\text{m}^2
\end{aligned}
$$

Volume $= 37.5 \times 10\,\text{m}^3 = 375\,\text{m}^3$

Section 7.2.1

1 The roundabout has gone through $4 \times 360° = 1440°$.

2

(a) $C\hat{A}B$ or $\angle CAB$

(b) $B\hat{C}A$ or $\angle BCA$

(c) γ or $\angle DAC$

(d) δ or $A\hat{C}D$

3

(a) $38°$: acute

(b) $120°$: obtuse

(c) $90°$: right angle

(d) $45°$: acute

(e) $155°$: obtuse

Section 7.2.2

1

(a) Each of the four angles is $360° \div 4 = 90°$.

(b) The two upper angles are both $180° \div 2 = 90°$, and the lower angle is $180°$.

(c) Each of the six angles is $360° \div 6 = 60°$.

(d) Each of the twenty angles is $360° \div 20 = 18°$.

(e) The acute angle between the hands is $360° \div 12 = 30°$; the reflex angle is $360° - 30° = 330°$.

(f) Each of the three angles $360° \div 3 = 120°$.

2

(a) $\gamma = 180° - 60° - 90° = 30°$

(b) $\delta = 360° - 60° - 60° - 130° = 110°$

3

(a) $130°, 50°, 130°$.

(b) $120°, 60°, 120°$.

(c) $90°, 90°, 90°$.

Section 7.2.3

1

(a) $\alpha = 180° - 130° - 30° = 20°$

(b) $\beta = 180° - 90° - 25° = 65°$
$\theta = 180° - \beta = 115°$
$\alpha = 180° - 50° - \theta = 15°$

2

(a) $\alpha = \beta$, so $60° + 2\alpha = 180°$.
Therefore $2\alpha = 120°$, and $\alpha = 60°$.

(b) $\gamma = \delta$, so $90° + 2\gamma = 180°$.
Therefore $2\gamma = 90°$, and $\gamma = 45°$.

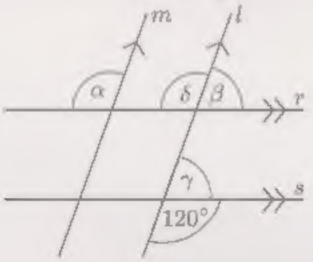
3 (a) and (b) are congruent; (a), (b), and (c) are similar. (d) and (e) are similar. (f) and (g) are congruent; (f), (g) and (h) are similar.

Section 7.2.4

1

(a) $\beta + 60° = 180°$, so $\beta = 120°$.

(b) $\alpha = \beta$ (corresponding angles), so $\alpha = 120°$.



$\gamma + 120° = 180°$, so $\gamma = 60°$.
$\gamma = \beta$ (corresponding angles), so $\beta = 60°$.
$\delta + \beta = 180°$, so $\delta = 120°$.
$\alpha = \delta$ (corresponding angles), so $\alpha = 120°$.

(c) $\alpha = 45°$ (alternate angles).
$\beta = 55°$ (alternate angles).

(d) $\alpha = B\widehat{A}C = 30°$ (alternate angles).
$B\widehat{A}C + C\widehat{B}A + A\widehat{C}B = 180°$ (angle sum of triangle), so $A\widehat{C}B = 180° - B\widehat{A}C - C\widehat{B}A$
$= 180° - 30° - 70° = 80°$.
$A\widehat{C}B + \alpha + \beta = 180°$ (angle on a straight line), so $\beta = 180° - A\widehat{C}B - \alpha$
$= 180° - 80° - 30° = 70°$.

(e) $C\widehat{A}B = A\widehat{C}D = 40°$ (alternate angles).
$C\widehat{A}B + \alpha + 80° = 180°$ (angle sum of ΔABC), so $\alpha = 180° - 40° - 80° = 60°$.
$D\widehat{A}C = B\widehat{C}A = 80°$ (alternate angles).
$D\widehat{A}C + \beta + 40° = 180°$ (angle sum of ΔACD), so $\beta = 180° - 80° - 40° = 60°$.

2

(a) Yes, since $C\widehat{B}A = 180° - 100° = 80°$ and so the corresponding angles $C\widehat{B}A$ and $E\widehat{D}B$ are equal.

(b) No, since $X\widehat{T}Y = 180° - 70° - 80° = 30°$ and so the corresponding angles $X\widehat{T}Y$ and $T\widehat{S}Z$ are not equal.

Acknowledgements

HMSO: from *Household Survey* tables on pages 14 (bottom), 15 (middle), 16, charts on pages 18 (top), 18 (bottom), 19 (top), 23 (top), 23 (bottom), and graphs on pages 36 (bottom), 38; from *Social Trends* charts on pages 6 (middle), 24 (bottom) and graph on page 7; from *Rent Rebates* table on page 6.

Consumer Association: from *Which?* November 1977 chart on page 19 (bottom).

Enterprise Holidays: from *Enterprise Summer '80 Brochure* table on page 13.

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1.7